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Random and periodic operators in dimension 1: Decorrelation estimates in spectral statistics and resonances

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Résumé

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Cette thèse comporte deux parties qui correspondent à deux domaines distincts : les opérateurs aléatoires et les opérateurs périodiques en dimension 1.
Dans la première partie, nous prouvons une estimée de décorrélation pour un opérateur aléatoire avec désordre hors diagonal en dimension 1. En se servant de cette estimée, nous déduisons l'indépendance asymptotique des statistiques locales des valeurs propres près d'énergies distinctes positives dans le régime localisé. Finalement, nous donnons une démonstration alternative de l'estimée de décorrélation pour le modèle d'Anderson discret unidimensionnel.
La deuxième partie de cette thèse est dédiée à un problème de résonances pour l'opérateur de Schrödinger discret en dimension 1 avec potentiel périodique tronqué. Précisément, nous considérons l'opérateur sur la demi droite $H^{\mathbb{N}} = -\Delta + V$ et l'opérateur $H^{\mathbb{N}}_L = -\Delta + V\mathbb{1}_{[0,L]}$ agissant sur $\ell^2(\mathbb{N})$ avec la condition au bord de Dirichet en 0 et $L \in \mathbb{N}$ large. Nous étudions les résonances de $H^{\mathbb{N}}_L$ qui sont près du bord du spectre essentiel de $H^{\mathbb{N}}$ dans la limite $L \to +\infty$, donc compléter les résultats introduits dans [Klo] sur les résonances de l'opérateur $H^{\mathbb{N}}_L$.

Mots-clefs : Opérateurs de Schrödinger aléatoire, Estimées de décorrélation, Statistiques spectrales, Opérateurs de Schroödinger périodique, Résonances.

Abstract

This thesis consists of two parts: the random and periodic operators in dimension 1.
In the first part, we prove the decorrelation estimate for a 1D lattice Hamiltonian with off-diagonal disorder. Consequently, we deduce the asymptotic independence of the local level statistics near distinct positive energies in the localized regime. Finally, we revisit a known result on the decorrelation estimate for the 1D discrete Anderson model.
The second part of my thesis addresses questions on resonances for a 1D Schrödinger operators with truncated periodic potential. Precisely, we consider the half-line operator

$H^{\mathbb{N}} = -\Delta + V$ and $H_L^{\mathbb{N}} = -\Delta + V \mathbb{1}_{[0,L]}$ acting on $\ell^2(\mathbb{N})$ with Dirichlet boundary condition at 0. We describe the resonances of $H_L^{\mathbb{N}}$ near the boundary of the essential spectrum of $H^{\mathbb{N}}$ as $L \to +\infty$, hence complete the results introduced in [Klo] on the resonances of $H_L^{\mathbb{N}}$.

Keywords: Random Schrödinger operators, Decorrelation estimates, Spectral statistiques, Periodic Schrödinger operators, Resonances.

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CHAPTER 1

Introduction (Fr)

Contents

Cette thèse comporte deux parties qui correspondent à deux domaines distincts : les opérateurs aléatoires et les opérateurs périodiques en dimension 1. La première partie est dédiée à l'étude des estimées de décorrélation pour les opérateurs aléatoires discrets en dimension 1. Dans la deuxième partie, nous étudions un problème de résonances de l'opérateur de Schrödinger en dimension 1 avec potentiel périodique tronqué.

1.1 Estimées de décorrélation

La première partie s'inscrit dans le cadre de la théorie des opérateurs de Schrödinger aléatoires. Concrètement, on s'intéresse aux statistiques spectrales d'une famille de tels opérateurs discrets unidimensionnels i.e. définis sur l'espace $l^2(\mathbb{Z})$.

Définissons l'opérateur que nous allons étudier dans cette partie : Soit $\{\omega_n\}_{n\in\mathbb{Z}}$ une suite de variables aléatoires indépendantes identiquement distribuées (i.i.d.) sur un espace probabilisé complet $(\Omega, \mathcal{B}, \mathbb{P})$, à valeurs dans $\mathbb{R}$, de densité commune bornée et à support compact. On suppose de plus que $\omega_n \in [\alpha_0, \beta_0]$ pour tout $n \in \mathbb{Z}$ où $0 < \alpha_0 < \beta_0$. On définit ensuite l'opérateur aléatoire H_ω avec désordre hors diagonal en dimension 1 agissant sur $l^2(\mathbb{Z})$: pour $u = \{u(n)\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$, on définit

$$(H_\omega u)(n) = \omega_n(u(n) - u(n+1)) - \omega_{n-1}(u(n-1) - u(n)) \tag{1.1.1}$$

Il est facile de voir que $\{H_\omega\}_{\omega\in\Omega}$ est une famille ergodique des opérateurs auto-adjoints sur $l^2(\mathbb{Z})$. Donc, il existe un ensemble $\Sigma \in \mathbb{R}$ tel que $\sigma(H_\omega) = \Sigma$ $\mathbb{P}-$presque sûrement. On appelle Σ le *spectre presque sûr* de l'opérateur H_ω (voir Section 2.2 pour plus de détails). En d'autres termes, le spectre de H_ω est une quantité déterministe même si H_ω dépend de paramètres aléatoires. D'ailleurs, dans notre cas concret, on peut calculer explicitement le spectre presque sûr de l'opérateur H_ω pour trouver $\Sigma = [0, 4\beta_0]$ (voir [Mia11]).

Soit $\Lambda := [-L, L]$ un "cube" dans $\mathbb{Z}$, on définit l'opérateur du volume fini $H_\omega(\Lambda)$ comme la restriction de H_ω à Λ avec les conditions au bord périodique. Alors, $H_\omega(\Lambda)$ est une matrice symétrique de taille $|\Lambda| = 2L+1$. Donc, toutes ses valeurs propres sont réelles et on les note (en les répétant selon leur multiplicité) $E_1(\omega, \Lambda) \leqslant E_2(\omega, \Lambda) \leqslant \cdots \leqslant E_{|\Lambda|}(\omega, \Lambda)$. Notons que, bien que Λ soit un intervalle dans $\mathbb{Z}$, nous allons l'appeler cube et parler de "son volume" qui est juste son cardinal (le nombre de points dans Λ) ; en effet toutes les notations basiques que nous allons introduire ci-après sont applicables pour les opérateurs aléatoires discrets ou continus en dimension quelconque.

Maintenant, nous sommes prêt à définir une quantité de base de la théorie des opérateurs de Schrödinger aléatoires, la *densité d'états intégrée* de H_ω. La densité d'états intégrée

$N(E)$ en une énergie donnée E est la limite quand $|\Lambda|$ tend vers l'infini du nombre de valeurs propres de $H_\omega(\Lambda)$ plus petites que E comptées avec multiplicité divisé par le volume du cube $|\Lambda|$. Autrement, la densité d'états intégrée mesure le nombre moyen de niveaux d'énergie situés en-dessous l'énergie donnée E par unité de volume dans la limite $|\Lambda| \to +\infty$.

$$N(E) := \lim_{|\Lambda|\to+\infty} \frac{\#\{\text{valeurs propres de } H_\omega(\Lambda) \leq E\}}{|\Lambda|} \text{ pour presque partout } E.$$

Il est bien connu que, pour le modèle (1.1.1), la densité d'états intégrée est bien définie partout sur $\mathbb{R}$ et absolument continue par rapport à la mesure de Lebesgue (voir Section 3.1). Nous pouvons alors définir *la densité d'états* $\nu(E)$ de H_ω comme la dérivée de la densité d'états intégrée $N(E)$. D'ailleurs, le support de $N(E)$ coïncide le spectre presque sûr Σ.

L'existence de $N(E)$ implique que le nombre de valeurs propres dans un intervalle $I = [a, b]$ borné est approximativement égale à $|\Lambda|N(I)$ où $N(I) := N(b) - N(a)$ quand $|\Lambda|$ est suffisamment grand. Il en résulte que l'espacement moyen entre les valeurs propres de $H_\omega(\Lambda)$ (la distance moyenne de $E_{j+1}(\omega,\Lambda) - E_j(\omega,\Lambda)$) quand $|\Lambda|$ grand est de taille $|\Lambda|^{-1}$. Donc, les valeurs propres de $H_\omega(\Lambda)$ sont très proches les unes les autres quand $|\Lambda| \to +\infty$.

Naturellement, on voudrait comprendre mieux comment les valeurs propres de $H_\omega(\Lambda)$ sont distribuées sur l'axe réel à la limite $|\Lambda| \to +\infty$ en notant qu'elles sont des variables aléatoires dépendantes les unes des autres. Ceci nous amène à étudier les statistiques spectrales *locales* près d'une énergie de référence E.

Dans le cadre de cette thèse, nous nous restreignons aux statistiques spectrales dans le régime localisé. Tout d'abord, on définit le régime localisé $I \in \Sigma$ de l'opérateur H_ω comme la région où le spectre de H_ω est discret et les vecteurs propres associés décroissent exponentiellement vers 0 à l'infini. En dimension 1, on peut prendre $I = \Sigma$ (voir Section 3.3). Puis, on prend une énergie $E > 0$ dans I tel que $\nu(E) > 0$ où ν est la densité d'état de H_ω (on dit que E appartient au «bulk » du spectre de H_ω). Alors, la statistique des niveaux locaux au voisinage de E est le processus ponctuel suivant

$$\Xi(\xi, E, \omega, \Lambda) := \sum_{n=1}^{|\Lambda|} \delta_{\xi_n}(E, \omega, \Lambda)(\xi) \tag{1.1.2}$$

où

$$\xi_n(E, \omega, \Lambda) = |\Lambda|\nu(E)(E_n(\omega, \Lambda) - E).$$

Un de résultats les plus remarquables dans les statistiques spectrales des opérateurs aléatoires, c'est que, localement, les valeurs propres d'un opérateur aléatoire dans le régime localisé sont distribuées de façon poissonnienne [Mol81, Min96, GK14], c'est-à-dire, le processus $\Xi(\xi, E, \omega, \Lambda)$ converge faiblement vers un processus de Poisson. Un tel résultat indique l'absence de répulsion de niveaux d'énergie qui est une caractéristique du régime localisé.

Nous présentons maintenant la question centrale dans cette partie de ma thèse, *les estimées de décorrélation (D)*. Comme nous venons de le voir, les valeurs propres d'un opérateurs aléatoires près d'une énergie fixée E dans le régime localisé sont distribuées sur l'axe réel comme les points d'un processus de Poisson. Alors, si l'on considère deux énergies de références $E \neq E'$ dans le régime localisé au lieu d'une seul énergie et on prend les deux processus ponctuels correspondants $\Xi(\xi, E, \omega, \Lambda), \Xi(\xi, E', \omega, \Lambda)$. Ils convergent chacun vers un processus de Poisson. Il est naturel de se demander si les limites ainsi obtenues sont indépendantes ?
Cette question a été résolue affirmativement pour le modèle d'Anderson discret dans [Klo11]. Un des ingrédients cruciaux pour démontrer ce résultat dans le cas du modèle d'Anderson discret sont des inégalités qui s'appellent les **estimées de décorrélation**.

Dans cette partie de ma thèse, nous étudions ce problème pour l'opérateur avec désordre hors diagonal (1.1.1). Concrètement, nous démontrons l'estimée de décorrélation suivante pour les valeurs propres proches de deux énergies positives, distinctes dans le régime localisé du modèle (1.1.1) :

Theorem 1.1.1. *[Tri14, Theorem 1.2] Soit E, E' deux énergies positives, distinctes dans le régime localisé. Soit $\beta \in (1/2, 1)$ et $\alpha \in (0,1)$. Alors, pour tout $c > 0$, il existe $C > 0$ t.q., pour tout L grand et $cL^\alpha \leqslant l \leqslant L^\alpha/c$, on a*

$$\mathbb{P}\left(\left\{\begin{array}{l}\sigma(H_\omega(\Lambda_l)) \cap (E + L^{-1}(-1,1)) \neq \emptyset \\ \sigma(H_\omega(\Lambda_l)) \cap (E' + L^{-1}(-1,1)) \neq \emptyset\end{array}\right\}\right) = o\left(\frac{l}{L}\right). \tag{1.1.3}$$

Grâce à l'estimée de décorrélation (1.1.3) ci-dessus, on déduit que les limites des statistiques locales des valeurs propres du modèle (1.1.1) près deux énergies positives, distinctes sont indépendantes. De plus, on montre qu'un tel résultat reste vrai pour *non seulement deux mais aussi n énergies positives et distinctes* quelque soit $n \geq 2$.

Theorem 1.1.2. *[Tri14, Theorem 5.1] Soit $n \geqslant 2$ et $\{E_i\}_{1 \leqslant i \leqslant n}$ une suite finie d'énergies positives, distinctes dans le régime localisé t.q. $\nu(E_i) > 0$ pour tout $1 \leq i \leq n$.*
Alors, quand $|\Lambda| \to +\infty$, les n processus ponctuels $(\Xi(\xi, E_i, \omega, \Lambda))_{1 \leq i \leq n}$ définis comme dans (1.1.2) convergent faiblement vers n processus de Poisson indépendants.

Au fait, Théorème 1.1.2 est universel au sens que, *quelque soit le modèle aléatoire $\mathbb{Z}^d$-ergodique en dimension d quelconque, si l'estimée de Wegner (W), l'estimée de Minami (M) (c.f. Chapitre 3) et l'estimée de décorrélations sont connus pour ce modèle là, on a l'indépendance asymptotique des statistiques locales des valeurs propres près des énergies distinctes positives dans le régime localisé*. Donc, la seule raison pourquoi on restreint sur un modèle aléatoire unidimensionnel vient de la difficulté de démontrer une estimée de décorrélation en dimension supérieure.

D'ailleurs, notre stratégie pour prouver l'estimée de décorrélation (1.1.3) est adaptable pour le modèle d'Anderson discret aussi. Dans Section 4.5, en appliquant cette stratégie, nous obtenons une démonstration alternative pour l'estimée de décorrélation du modèle d'Anderson discret en dimension 1.

Finalement, nous voulons dire quelques mots sur le cas multi-dimensionnel. La seule difficulté qui nous empêche d'étendre nos résultats au cas multidimensionnel, c'est que, dans la preuve du Théorème 1.1.1 nous nous sommes servis forcement une estimée pour les composantes de chaque vecteur propre normalisé de $H_\omega(\Lambda)$ et une telle estimée n'est pas du tout connue en dimension supérieure. Soit u un vecteur propre normalisé de l'opérateur du volume fini $H_\omega(\Lambda)$. Alors, $u \in \mathbb{R}^{|\Lambda|}$ et

$$\|u\|^2_{l^2(\mathbb{R}^{|\Lambda|})} = \sum_{j\in\Lambda} |u_j|^2 = 1.$$

En dimension 1, grâce aux matrices de transfert, il est très facile de démontrer qu'il y a beaucoup composantes u_j de u qui ne sont pas trop petites (voir Lemma 4.2.2 du chapitre 4). Précisement, soit $\beta \in (1/2, 1)$. Dans un intervalle $\Lambda \subset \mathbb{Z}$, il existe au moins un sous-intervalle Λ_1 de taille $|\Lambda|^\beta$ tel que $|u_j| \geq e^{-c|\Lambda|^\beta}$ pour tout $j \in \Lambda_1$.

Un tel résultat est classique pour les opérateurs aléatoires en dimension 1 mais est encore un défi en dimension supérieure. On peut le considérer comme *une version quantitative du principe de continuation unique dans le cas discret unidimensionnel.*

Ceci nous rappelle d'un article remarquable de Bourgain et Kenig sur la localisation du modèle de Bernoulli continu en dimension quelconque[BK05]. Pour démontrer la localisation presque sûr au bord du spectre de cet opérateur, ils ont prouvé une *version quantitative du principe de continuation unique* pour les solutions de l'équation de Schrödinger stationnaire $H_B\xi = E\xi$ où $H_B = -\Delta + V_\omega$ est l'opérateur de Schrödinger sur $L^2(\mathbb{R}^d)$ et

$$V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \varepsilon_j \varphi(x-j) \text{ où } \varepsilon_j \text{ sont v.a. Bernoulli i.i.d.}$$

La version classique du principe de continuation unique nous dit que si l'une solution stationnaire ξ de H_B s'annule dans un ouvert de $\mathbb{R}^d$, elle s'annule partout. La version quantitative de Bourgain et Kenig donne beaucoup plus d'infos sur la taille de la fonction ξ dans une boule d'unité :

$$\int_{B(j,1)} |\xi(x)|^2 dx \gtrsim e^{-c|j|^{4/3}\ln|j|} \text{ pour } j \in \mathbb{Z} \text{ et } |j| \to +\infty.$$

D'ailleurs, les auteurs dans [BK05] ont fait un commentaire qu'une inégalité pareille pour la version discrète du modèle Bernoulli est encore inconnue et la question sur la localisation pour le modèle de Bernoulli discret est donc toujours ouverte.

1.2 Résonances quantiques

La deuxième partie de ma thèse est dédiée à l'étude des résonances de l'opérateur de Schrödinger avec potentiel périodique tronqué.
Soit V un potentiel périodique et $-\Delta$ le Laplacien discret sur $l^2(\mathbb{Z})$. On définit l'opérateur de Schrödinger $H^{\mathbb{Z}} := -\Delta + V$ en dimension 1:

$$(H^{\mathbb{Z}}u)(n) = ((-\Delta + V)u)(n) = u(n-1) + u(n+1) + V(n)u(n), \quad \forall n \in \mathbb{Z}. \tag{1.2.1}$$

Par ailleurs, on considère aussi l'opérateur $H^{\mathbb{N}} := -\Delta + V$ agissant sur $l^2(\mathbb{N})$ avec condition au bord de Dirichlet en 0.
Soit $\Sigma_{\mathbb{Z}}$ le spectre de $H^{\mathbb{Z}}$ et $\Sigma_{\mathbb{N}}$ le spectre de $H^{\mathbb{N}}$. Voici la description du spectre de $H^{\bullet}$ où $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$:

- $\Sigma_{\mathbb{Z}} = \bigcup_{j=1}^{q} B_q$ avec $q \leq p$ et $B_q = [c_q, d_q]$; le spectre $\Sigma_{\mathbb{Z}}$ est absolument continu (a.c.) et la résolution spectrale peut-être obtenue par la décomposition de Bloch-Floquet.
- $\Sigma_{\mathbb{N}} = \Sigma_{\mathbb{Z}} \cup \{v_j\}_{j=1}^m$ où $\Sigma_{\mathbb{Z}}$ est le spectre a.c. de $H^{\mathbb{N}}$ et $\{v_j\}_{j=1}^m$ sont des valeurs propres simples associées à des vecteurs propres exponentiellement décroissants.

Soit L large, on définit:

$$H_L^{\mathbb{N}} := -\Delta + V\mathbb{1}_{[0,L]} \text{ sur } l^2(\mathbb{N}) \text{ avec condition au bord de Dirichlet en } 0.$$

Il est facile de justifier que l'opérateur $H_L^{\mathbb{N}}$ est auto-adjoints. Donc, la résolvante $z \in \mathbb{C}^+ \mapsto (z - H_L^{\mathbb{N}})^{-1}$ est bien définie sur $\mathbb{C}^+$. De plus, on peut démontrer que (c.f. [Klo, Theorem 1.1]), $(z - H_L^{\mathbb{N}})^{-1}$ admet un prolongement méromorphe de $\mathbb{C}^+$ à $\mathbb{C}\backslash((-\infty, -2] \cup [2, +\infty))$ à valeurs dans l'ensemble d'opérateurs auto-adjoints de l^2_{comp} à l^2_{loc}.

D'ailleurs, l'ensemble de pôles d'un tel prolongement méromorphe appartient au demi-plan inférieur $\{ImE < 0\}$ et son cardinal est au plus L.
*Les **résonances** de $H_L^{\mathbb{N}}$ sont alors définies comme étant les pôles du prolongement ci-dessus.*
La distribution et l'asymptotique de résonances de $H_L^{\mathbb{N}}$ quand $L \to +\infty$ ont été étudiées intensivement dans l'article [Klo]. Tous résultats dans [Klo] sont prouvés sous l'hypothèse que les parties réelles de résonances sont éloignées du bord du spectre $\Sigma_{\mathbb{Z}}$ et ± 2, le bord du spectre essentiel du Laplacien discret libre. Précisément, dans [Klo], l'auteur a étudié les résonances dans le domaine $I - i\mathbb{R}^+$ où I est un intervalle compact soit à l'intérieur soit à l'extérieur du spectre $\Sigma_{\mathbb{Z}}$.

Dans cette partie de ma thèse, nous nous intéressons aux résonances de $H_L^{\mathbb{N}}$ dont les parties réelles sont près du bord de $\Sigma_{\mathbb{Z}}$ i.e., on cherche les résonances dans le domaine $I - i\mathbb{R}^+$ où l'intervalle I contient les points au bord de $\Sigma_{\mathbb{Z}}$ et la taille de I est petite.

1.2.1 L'équation de résonance et la stratégie pour étudier les résonances

Soit $L > 0$ et H_L l'opérateur $H_L^{\mathbb{N}}$ restraint sur l'intervalle $[0, L]$ avec les conditions au bord Dirichlet à L. On définit

- $(\lambda_k)_{0 \leq k \leq L}$ la suite croissante des valeurs propres de H_L.
- $a_k = |\varphi_k(L)|^2$ où $\varphi_k = (\varphi_k(n))_{0 \leq n \leq L}$ est un vecteur propre normalisé associé à λ_k.

Alors, l'équation de résonance pour l'opérateur $H_L^{\mathbb{N}}$ est la suivante (c.f. [Klo, Théorème 2.1]):

$$S_L(E) := \sum_{k=0}^{L} \frac{a_k}{\lambda_k - E} = -e^{-i\theta(E)}, \qquad E = 2\cos\theta(E), \tag{1.2.2}$$

où l'on choisit la détermination de $\theta(E)$ t.q. $\text{Im}\theta(E) > 0$ et $\text{Re}\theta(E) \in (-\pi, 0)$ quand $\text{Im}E > 0$.
Soit E_0 un point au bord du spectre $\Sigma_{\mathbb{Z}}$ et $E_0 \in (-2, 2)$. Notons que ± 2 sont les points du bord du spectre essentiel du Laplacien libre ainsi que les points tournants de la fonction $\theta(E)$ dans l'équation (1.2.2). Sans perte de généralité, nous supposons que $E_0 \in B_i = [E_0, E_1] \subset \Sigma_{\mathbb{Z}}$. Puis, nous numérotons les valeurs propres $\lambda_k \in B_i$ et a_k correspondants comme $(\lambda_\ell^i)_\ell, (a_\ell^i)_\ell$ où $0 \leq \ell \leq n_i$ (la numérotation locale par rapport à la bande B_i).
Soit $\varepsilon_1 > 0$ un nombre fixé et suffisamment petit t.q. l'intervalle $I = I_{\varepsilon_1} = [E_0, E_0 + \varepsilon_1] \in (-2, 2) \cap \Sigma_{\mathbb{Z}}$ et $E_0 + \varepsilon_1 \in \overset{\circ}{\Sigma}_{\mathbb{Z}}$. Remarque que, quand $\text{Re}E > \varepsilon_1$, la description de résonance

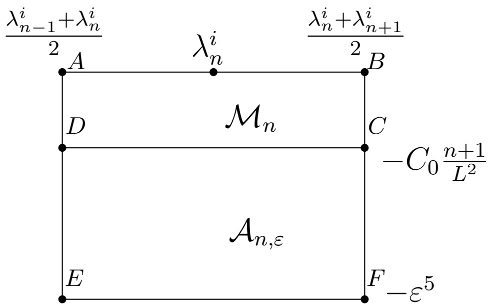


Figure 1.1: Rectangle $\mathcal{B}_{n,\varepsilon}$

peut-être est trouvé dans [Klo]. Il est suffisant de "résoudre" l'équation (1.2.2) dans le rectangle $R_E := I_{\varepsilon_1} - i[0, \varepsilon_2]$ où $\varepsilon_1, \varepsilon_2$ sont deux petites, positives constantes. En dehors de ce rectangle, tous les résulats dans [Klo] sont encore valides. Dans cette partie de ma thèse, nous utilisons toujours $\varepsilon_1 \asymp \varepsilon^2$ et $\varepsilon_2 \asymp \varepsilon^5$ où ε est un petit paramètre fixé.
Pour étudier les résonances près du point $E_0 \in \partial\Sigma_{\mathbb{Z}}$, on va se servir des asymptotiques de deux paramètres λ_k et a_k pour simplifier l'équation (1.2.2) et on utilise une équation approximative pour obtenir l'existence, l'unicité et les asymptotiques de résonances.
Notons que, si λ^i_n est une valeur propre près de E_0 et $\lambda^i_n \in B_i = [E_0, E_1]$, on a $|\lambda^i_n - E_0| \asymp \frac{n^2}{L^2}$. Dans [Klo], l'auteur a indiqué qu'il y a deux comportements possibles pour a^i_n associée à une valeur propre près E_0: soit a^i_n est de taille $\frac{1}{L}$, soit a^i_n devient beaucoup plus petit, sa taille est $\frac{|\lambda^i_n - E_0|}{L}$. De plus, le dernier cas est générique. Concernant ces deux cas, on propose deux approches différentes pour étudier les résonances.

1.3 Résonances dans le cas générique

Dans cette section, nous supposons que $a_k \asymp \frac{|\lambda_k - E_0|}{L}$ quand λ_k est près du point $E_0 \in \partial\Sigma_{\mathbb{Z}} \cap (-2, 2)$. Nous allons étudier l'équation de résonance (1.2.2) dans le rectangle $[E_0, E_0 + \varepsilon^2] - i[0, \varepsilon^5]$ où $\varepsilon > 0$ est petit.
L'approche utilisée pour étudier les résonances dans le cas présent est la suivante. Pour chaque $0 \le n \le \varepsilon L/C_1$ avec $C_1 > 0$ grand, nous cherchons les résonances dans le rectangle $\mathcal{B}_{n,\varepsilon} = \left[\frac{\lambda^i_{n-1}+\lambda^i_n}{2}, \frac{\lambda^i_n+\lambda^i_{n+1}}{2}\right] - i\left[0, \varepsilon^5\right]$ avec la convention $\lambda^i_{-1} = 2E_0 - \lambda_0$.
Tout d'abord, nous montrons que, $|\mathrm{Im} S_L(E)|$ est très petite dans le rectangle $\mathcal{A}_{n,\varepsilon} = \left[\frac{\lambda^i_{n-1}+\lambda^i_n}{2}, \frac{\lambda^i_n+\lambda^i_{n+1}}{2}\right] - i\left[C_0\frac{n+1}{L^2}, \varepsilon^5\right]$ où la constante$C_0 > 0$ est grande (voir Figure 1.1).

Donc, il n'y a pas de résonances dans $\mathcal{A}_{n,\varepsilon}$. Puis, dans $\mathcal{M}_n$, le complément de $\mathcal{A}_{n,\varepsilon}$ dans $\mathcal{B}_{n,\varepsilon}$, nous remplaçons l'équation de résonances (1.2.2) par une équation beaucoup plus simple en appliquant le théorème de Rouché. Ceci nous permet de démontrer l'existence et l'unicité d'une résonance, disons z_n, dans chaque $\mathcal{M}_n$. Finalement, nous démontrons la régularité des paramètres spectraux a_k, λ_k près E_0 et nous nous en servons pour obtenir les asymptotiques de z_n et sa partie imaginaire.
Voici le théorème principal qui décrit les résonances près de E_0 dans le cas générique.

Theorem 1.3.1. *[Tria, Theorem 1.1] Soit $E_0 \in (-2,2)$ l'extrémité gauche d'une bande B_i de $\Sigma_{\mathbb{Z}}$. On numérote les paramètres spectraux λ_k et a_k dans B_i comme $(\lambda^i_\ell)_\ell, (a^i_\ell)_\ell$ où $0 \le \ell \le n_i$ (la numérotation locale par rapport à la bande B_i).*
Soit $I = [E_0, E_0 + \varepsilon_1]$ et $\mathcal{D} = [E_0, E_0+\varepsilon_1] - i[0, \varepsilon_2]$ où $\varepsilon_1 \asymp \varepsilon^2$ et $\varepsilon_2 \asymp \varepsilon^5$ avec $\varepsilon > 0$ petit. Alors,

1. *Pour chaque valeur propre $\lambda^i_n \in I$ de H_L, il y a une et une seule résonance z_n dans $\mathcal{B}_{n,\varepsilon} = \left[\frac{\lambda^i_{n-1}+\lambda^i_n}{2}, \frac{\lambda^i_n+\lambda^i_{n+1}}{2}\right] - i\left[0, \varepsilon^5\right]$ avec la convention $\lambda^i_{-1} := 2E_0 - \lambda_0$. De plus, $z_n \in \mathcal{M}_n = \left[\frac{\lambda^i_{n-1}+\lambda^i_n}{2}, \frac{\lambda^i_n+\lambda^i_{n+1}}{2}\right] - i\left[0, C_0\frac{n+1}{L^2}\right]$ avec $C_0 > 0$ large. Par ailleurs, il n'y a pas de résonances dans le rectangle $[E_0 - \varepsilon, E_0] - i\left[0, C_0\frac{n+1}{L^2}\right]$.*
2. *On définit $S^i_{n,L}(E) = S_L(E) - \frac{a^i_n}{\lambda^i_n - E}$ et $\alpha_n = S^i_{n,L}(\lambda^i_n) + e^{-i\theta(\lambda^i_n)}$. Alors, il existe $c_0 > 0$ t.q. $c_0 \le |\alpha_n| \lesssim \frac{1}{\varepsilon^2}$ et*
$$z_n = \lambda^i_n + \frac{a^i_n}{\alpha_n} + O\left(\frac{(n+1)^4}{L^5|\alpha_n|^3}\right).$$
3. *Imz_n satisfait*
$$Imz_n = \frac{a^i_n \sin(\theta(\lambda^i_n))}{|\alpha_n|^2} + O\left(\frac{(n+1)^4}{L^5|\alpha_n|^3}\right).$$
Par conséquent, il existe une constante $C > 0$ t.q. $\frac{\varepsilon^4(n+1)^2}{CL^3} \le |Imz_n| \le C\frac{(n+1)^2}{L^3}$.

Comme nous venons de le voir dans le théorème ci-dessus, chaque valeur proper $\lambda^i_n \in \mathring{\Sigma}_{\mathbb{Z}}$ près $E_0 \in \partial\Sigma_{\mathbb{Z}}$, génère une et une seule résonance z_n de $H^{\mathbb{N}}_L$ et $|\mathrm{Im} z_n| \asymp \frac{n^2}{L^3}$. Donc, quand $n \asymp \varepsilon L$ (loin du bord de $\Sigma_{\mathbb{Z}}$), la taille de $|\mathrm{Im} z_n|$ est $\frac{1}{L}$ et on retrouve les résultats obtenus dans [Klo]. Quand n est petit (près du bord), $|\mathrm{Im} z_n|$ devient beaucoup plus petit. Sa taille varie entre $\frac{1}{L^3}$ et $\frac{1}{L}$.
Finalement, nous tournons notre attention vers les résonances en dessous $\mathbb{R}\backslash\Sigma_{\mathbb{N}}$. Rappelons que $\Sigma_{\mathbb{N}}$, le spectre de $H^{\mathbb{N}}$, est la réunion de $\Sigma_{\mathbb{Z}}$ et l'ensemble des valeurs propres simples, isolées de $H^{\mathbb{N}}$.

Soit I un interval compact dans $(-2,2)$ et $I \subset \mathbb{R}\backslash\Sigma_{\mathbb{N}}$. Alors, d'après [Klo, Théorème 1.2], il existe une constante $c > 0$ t.q. $H_L^{\mathbb{N}}$ n'a pas de résonances dans le rectangle $I - i[0,c]$. En fait, on peut démontrer que ce résultat reste encore vrai même si l'intervalle I touche le bord du spectre $\Sigma_{\mathbb{Z}}$.

Theorem 1.3.2. *[Tria, Theorem 1.2] Soit $E_0 \in (-2,2)$ l'extrémité gauche d'une bande B_i de $\Sigma_{\mathbb{Z}}$. Soit $L \in \mathbb{N}^*$ large. Alors, $H_L^{\mathbb{N}}$ n'a pas de résonances dans le rectangle $[E_0-\varepsilon, E_0] - i[0,\varepsilon^5]$ quand ε est suffisamment petit.*

1.4 Résonances dans le cas non-générique

Dans cette section, on étudie les solutions de l'équation de résonance (1.2.2) sous l'hypothèse que $a_k \asymp \frac{1}{L}$ quand la valeur propre λ_k associée est près du point $E_0 \in \partial\Sigma_{\mathbb{Z}} \cap (-2,2)$. Sans perte de généralité, nous supposons que E_0 est l'extrémité gauche de la bande $B_i = [E_0, E_1]$ de $\Sigma_{\mathbb{Z}}$.
En posant $z := L^2(E-E_0)$, l'équation de résonance (1.2.2) peut s'écrire comme

$$f_L(z) := \sum_{k=0}^{L} \frac{\tilde{a}_k}{\tilde{\lambda}_k - z} = -\frac{1}{L} e^{-i\theta(E)}. \tag{1.4.1}$$

où $\tilde{a}_k := La_k$, $\tilde{\lambda}_k := L^2(\lambda_k - E_0)$.
Rappelons que $|\lambda_n^i - E_0|$ est de taille $\frac{n^2}{L^2}$ si λ_n^i est près de E_0 et $\lambda_n^i \in B_i$ (la numérotation locale par rapport à la bande B_i). Donc, près de E_0, $\tilde{\lambda}_n^i \asymp n^2$ et $\tilde{a}_n^i \asymp 1$. Autrement, les paramètres $\tilde{a}_k, \tilde{\lambda}_k$ sont de taille constante près de E_0. Le but maintenant est d'étudier les résonances rééchelonnées z dans le rectangle $\tilde{\mathcal{D}} = [0, \varepsilon_1 L^2] - i[0, \varepsilon_2 L^2]$ où $\varepsilon_1 \asymp \varepsilon^2$ et $\varepsilon_2 \asymp \varepsilon^5$. Soit $(\lambda_\ell^i)_\ell$ les valeurs propres de H_L dans B_i. Nous écrivons $\tilde{\mathcal{D}}$ comme la réunion de $\mathcal{D}_n^i = [\tilde{\lambda}_n^i, \tilde{\lambda}_{n+1}^i] - i[0,\varepsilon_2 L^2]$ avec $0 \le n \lesssim \varepsilon L$ et le rectangle $\mathcal{R}^i = [0, \tilde{\lambda}_0^i] - i[0, \varepsilon_2 L^2]$. Ensuite, nous étudions l'existence et l'unicité de résonances dans chaque rectangle $\mathcal{D}_n^i$ et dans le rectangle $\mathcal{R}^i$.

1.4.1 Domaine qui ne contient pas de résonances

On observe que, $f_L(z)$ est une fonction méromorphe sur $\mathbb{C}$ avec des poles $\{\tilde{\lambda}_k\}$. Donc, $|f(z)|$ doit être grand près de ces poles là. Dans chaque rectangle $\mathcal{D}_n^i = [\tilde{\lambda}_n^i, \tilde{\lambda}_{n+1}^i] - i[0, \varepsilon^5 L^2]$ où $n \in [0, \varepsilon L/C_1]$ avec C_1 grand, f_L a deux pôles et dans $\mathcal{R}$, f_L a un seul pôle. On établit ici une version quantitative pour cette observation et on se servira de ce lemme plus tard pour décrire les régions ne contenant pas de résonances:

Lemma 1.4.1. *[Trib, Lemma 3.2] Soit $E_0 \in (-2, 2)$ l'extrémité gauche d'une bande B_i de $\Sigma_{\mathbb{Z}}$. Supposons que $(\lambda^i_\ell)_\ell$ avec $0 \le \ell \le n_i$ sont les valeurs propres de H_L dans B_i. Soit $I = [E_0, E_0 + \varepsilon_1] \subset B_i$ où $\varepsilon_1 \asymp \varepsilon^2$ avec $\varepsilon > 0$ petit.*
Pour chaque $0 \le n \le \varepsilon L/C_1$ avec $C_1 > 0$ grand, on définit

$$f_{n,L}(z) := \frac{\tilde{a}_n}{\tilde{\lambda}_n - z} + \frac{\tilde{a}_{n+1}}{\tilde{\lambda}_{n+1} - z}; \quad \tilde{f}_{n,L}(z) := f_L(z) - f_{n,L}(z)$$

où $z = L^2(E - E_0)$ avec $E \in I - i[0, \varepsilon^5]$.
Soit $\Delta_n := \frac{c_0(n+1)}{\kappa(\ln(n+1)+1)}$ où κ est une grande constante.
Alors,

- $|\tilde{f}_{n,L}(z)| \lesssim \frac{\ln(n+1)+1}{n+1}$ *pour tout* $z \in [\tilde{\lambda}^i_n, \tilde{\lambda}^i_{n+1}] + i\mathbb{R}$,
- $\tilde{f}'_{n,L}(z) \asymp \frac{1}{(n+1)^2}$ *si z est réel et* $z \in [\tilde{\lambda}^i_n, \tilde{\lambda}^i_{n+1}]$,
- $|Im\tilde{f}_{n,L}(z)| \lesssim \frac{|Imz|}{n^2}$.

Par conséquent, pour tout $z \in [\tilde{\lambda}^i_n, \tilde{\lambda}^i_n \pm \Delta_n] - i[0, \Delta_n]$,

$$|f_L(z)| \gtrsim \frac{1}{\Delta_n} \gtrsim \frac{1}{\varepsilon L}.$$

Note que, dans la définition de Δ_n, on choisit κ suffisamment grand pour que $\tilde{\lambda}^i_n - \Delta_n > 0$.

Le lemme ci-dessus montre qu'il n'y a pas de résonance dans $[\tilde{\lambda}^i_n, \tilde{\lambda}^i_n \pm \Delta_n] - i[0, \Delta_n]$ pour tout $0 \le n \lesssim \varepsilon L$. De plus, une autre région de non résonances sera obtenue en établissant quelques estimées sur $\mathrm{Im} f(z)$. Grosso modo, le lemme suivant décrit une région dans R_n où $|\mathrm{Im} f(z)|$ est grande par rapport à la valeur absolue de la partie imaginaire du terme à droite de (1.4.1):

Lemma 1.4.2. *[Trib, Lemma 3.3] On garde les notations et les hypothèses du lemme 1.4.1. Pour $1 \le n \le \varepsilon L/C_1$ avec $C_1 > 0$ grand, on pose $\mathcal{D}^i_n = [\tilde{\lambda}^i_n, \tilde{\lambda}^i_{n+1}] - i[0, \varepsilon^5 L^2]$ et $x_0 = L^2(\lambda^i_{n+1} - \lambda^i_n)$. Alors, dans $\mathcal{D}^i_n$, on a $|Imf_L(z)| \gtrsim \frac{1}{\varepsilon L}$ for all $\frac{x_0^2}{\varepsilon L} \le |Imz| \le \varepsilon^5 L^2$. Par ailleurs, l'assertion ci-dessus est encore valide dans le rectangle $[0, \tilde{\lambda}^i_1] - i\left[\frac{1}{\varepsilon L}, \varepsilon^5 L^2\right]$.*

En résumé, on obtient des zones de non résonances suivantes dépendantes de valeur de n:

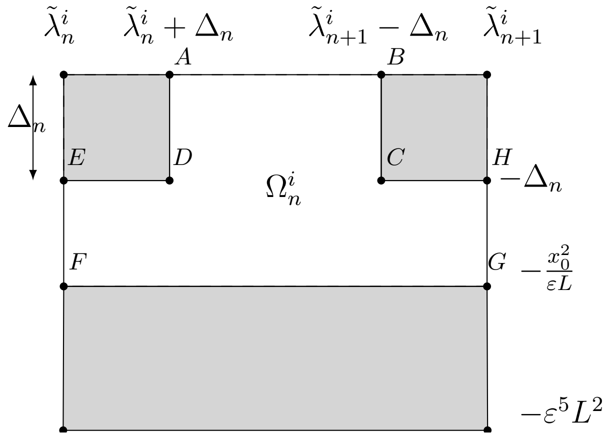


Figure 1.2: Zone ne contenant pas de résonances quand $\Delta_n < \frac{x_0^2}{\varepsilon L}$

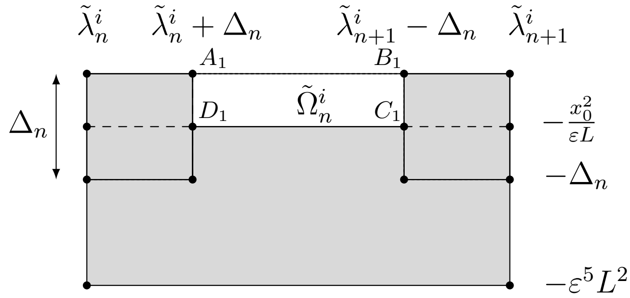


Figure 1.3: Zone ne contenant pas de résonances quand $\Delta_n \geq \frac{x_0^2}{\varepsilon L}$

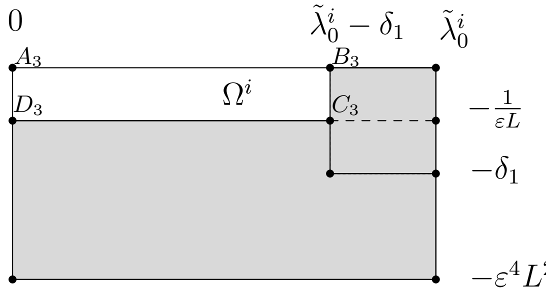


Figure 1.4: Zone ne contenant pas de résonances dans $\mathcal{R}^i := [0, \tilde{\lambda}_0^i] - i[0, L^2\varepsilon]$

1.4.2 Existence de résonances

Dans cette sous-section, on démontrera que, dans le domaine $\tilde{\Omega}_n^i = [\tilde{\lambda}_n^i + \Delta_n, \tilde{\lambda}_{n+1}^i - \Delta_n] - i\left[0, \frac{x_0^2}{\varepsilon L}\right]$ où $x_0 = L^2(\tilde{\lambda}_{n+1}^i - \tilde{\lambda}_n^i)$ correspondant au cas où n n'est pas trop grand ($n < \frac{\eta L}{\ln L}$) (c.f. Figure 1.3), il existe une et une seule résonance rééchelonnée $\tilde{z}_n$. Par contre, dans $\Omega^i = [0, \tilde{\lambda}_0^i - \delta_1] - i\left[0, \frac{1}{\varepsilon L}\right]$ où $\delta_1 > 0$ est petit, il n'y a pas de résonance.
Finalement, dans Ω_n^i où $n > \frac{\eta L}{\ln L}$, on peut démontrer l'existence de résonance. De plus, si le point $-\frac{e^{-i\theta(E_0)}}{L}$ appartient à $f_L(ABCD)$ où $ABCD = [\tilde{\lambda}_n^i + \Delta_n, \tilde{\lambda}_{n+1}^i - \Delta_n] - i[0, \Delta_n]$ (c.f. la figure 1.2), la résonance rééchelonnée est unique, disons z_n et $|z_n| \leq \Delta_n \lesssim \frac{n^2}{\varepsilon L}$.
Rappelons que, notre changement d'échelle est $z = L^2(E - E_0)$. Au milieu de chaque bande du spectre, les parties imaginaires de résonances sont de taille $\frac{1}{L}$ selon l'étude dans [Klo]. D'après nos résultats, quand on est près du bord de $\Sigma_{\mathbb{Z}}$, la largeur des résonances devient beaucoup plus petite.
Tout d'abord, nous donnons le résultat de l'existence et l'unicité de résonances dans $\tilde{\Omega}_n^i$:

Theorem 1.4.3. *[Trib, Theorem 4.6] Soit $n < \frac{\eta L}{\ln L}$ avec $\eta > 0$ petit. Soit $x_0 = \tilde{\lambda}_{n+1}^i - \tilde{\lambda}_n^i$ et $\tilde{\Omega}_n^i$ le rectangle $[\tilde{\lambda}_n^i + \Delta_n, \tilde{\lambda}_{n+1}^i - \Delta_n] - i\left[0, \frac{x_0^2}{\varepsilon L}\right]$.*
Alors, f_L est une bijection de $\tilde{\Omega}_n^i$ sur $f_L(\tilde{\Omega}_n^i)$ et $|f_L'(z)| \gtrsim \frac{1}{n^2}$. Par conséquent, il existe une et une seule résonance rééchelonnée $\tilde{z}_n$ in $\tilde{\Omega}_n^i$ et $|Im\tilde{z}_n| \lesssim \frac{n^2}{\varepsilon L}$.

En revanche, quand on est "trop proche" du bord du spectre $\Sigma_{\mathbb{Z}}$, il n'y a pas de résonance:

Theorem 1.4.4. *[Trib, Theorem 4.7] Soit $0 < \delta_1 < \tilde{\lambda}_0^i$ une petite constante. Soit Ω^i le rectangle $[0, \tilde{\lambda}_0^i - \delta_1] - i\left[0, \frac{1}{\varepsilon L}\right]$ (c.f. la figure 1.4).*
Alors, f_L est bijective de Ω^i sur $f_L(\Omega^i)$ et $|f_L'(z)| \geq c > 0$. De plus, $f_L(\Omega^i)$ ne contient pas le point $-\frac{e^{-i\theta(E_0)}}{L}$, donc, il n'y a pas de résonances dans Ω^i.

Finalement, on considère le domaine Ω_n^i où $n > \frac{\eta L}{\ln L}$:

Theorem 1.4.5. *[Trib, Theorem 4.5] Suppose que $n > \eta\frac{L}{\ln L}$ et $x_0 = \tilde{\lambda}_{n+1}^i - \tilde{\lambda}_n^i$.*
Soit Ω_n^i le complément de deux carrés $[\tilde{\lambda}_n^i, \tilde{\lambda}_n^i + \Delta_n] - i[0, \Delta_n]$ et $[\tilde{\lambda}_{n+1}^i, \tilde{\lambda}_{n+1}^i - \Delta_n] - i[0, \Delta_n]$ dans le rectangle $[\tilde{\lambda}_n^i, \tilde{\lambda}_{n+1}^i] - i\left[0, \frac{x_0^2}{\varepsilon L}\right]$ (le domaine ABCHGFED dans la figure 1.2).
Alors, il existe au moins une résonance rééchelonnée dans Ω_n^i. Par ailleurs, si $-\frac{1}{L}e^{-i\theta(E_0)}$ appartient au $A'B'C'D' = f_L(ABCD)$ où $ABCD = [\tilde{\lambda}_n^i + \Delta_n, \tilde{\lambda}_{n+1}^i - \Delta_n] - i[0, \Delta_n]$, il existe une et une seule résonance rééchelonnée z_n dans Ω_n^i et

$$|Imz_n| \leq \Delta_n = \frac{n}{\kappa \ln n} \asymp \frac{n}{\kappa \ln L} \lesssim \frac{n^2}{\varepsilon L}.$$

1.5 Questions ouvertes

1. Quant à la première partie, malgré notre effort, une estimée de décorrélation pour les modèles discrets en dimension supérieure reste encore un défi. Récemment, les éstimées de décorrélations pour certains autres modèles discrets et continus sont obtenus par la recherche de Shirley (c.f. [Shi15]). Nous pensons qu'il y a encore des questions intéressantes à étudier sur ce genre des estimées en particulier et les statistiques spectrales en général.
2. L'étude de résonances de l'opérateur discret associé à un potentiel périodique sur la droite entière est une question que nous poursuivons après cette thèse. Dans ce cas là, les valeurs de $|\varphi_k(L)|^2$ et $|\varphi_k(0)|^2$ vont jouer un rôle crucial.
   Finalement, nous voulons également voir ce qui se passe pour les résonances loins du bord du spectre $\Sigma_{\mathbb{Z}}$ mais près les points ± 2, les points tournants de la fonction $\theta(E)$ définie dans (1.2.2). Notons que, quand on est loin du bord de $\Sigma_{\mathbb{Z}}$, notre méthode ne fonctionne pas. Dans ce cas là, il faut utiliser et améliorer la méthode introduite par Klopp [Klo].

Cette thèse est organisé comme suit: La première partie de thèse comporte 4 chapitres. Le chapitre 2 est dédié à une introduction brève de la motivation et les notions importantes d'opérateurs aléatoires. Puis, nous introduisons la statistique spectrale dans le chapitre 3. Ensuite, nous parlons des estimées de décorrélations du modèle (1.1.1) dans le chapitre 4. Les quatre premières sections de ce chapitre font partie de mon article [Tri14] publié en 2014 sur Annales Henri Poincaré. Finalement, nous avons ajouté en plus la section 4.5 où nous démontrons une solution alternative de l'estimée de décorrélation du modèle d'Anderson discret en dimension 1.
Les trois dernier chapitres, les chapitres 5-7 sont consacrés à la deuxième partie de thèse sur les résonances. Dans le chapitre 5, nous faisons un résumé des résultats dans [Klo] sur résonances. Le chapitre 6 est réservé à notre contribution récente à l'étude de résonances dans le cas générique. Finalement, nous étudions le cas non-générique dans le chapitre 7. Les chapitres 6-7 font partie de deux articles [Tria] et [Trib].

CHAPTER 2

Overview on Random Operators

Contents

2.1 Where do random operators come from?

Let's talk a little bit about physics (c.f. [Luc92], [Kir08]). Assume that we want to study how an electron propagates in a metal tube which **contains impurities**. At sufficiently low temperatures, the inelastic electron-phonon interactions are negligible. Then, only elastic collisions which are either between electrons or between an electron and the potential of the lattice are taken into account.

Neglecting the effect of Coulomb interactions between electrons, we are led to consider the following stationary Schrödinger equation in $L^2(\mathbb{R}^d)$ for one electron:

$$Hu := (-\Delta + V)u = Eu \text{ for } u \in L^2(\mathbb{R}^d), d \geq 1 \tag{2.1.1}$$

where $-\Delta$ is the Laplacian operator and V is the potential representing interactions between the electron and atoms of the lattice.

In the case of perfect crystal (without impurities), the atoms or nuclei are distributed on a periodic lattice, say $\mathbb{Z}^d$, in a regular way. If we assume that an electron at the point $x \in \mathbb{R}^d$ feels a potential $qf(x-i)$ due to an atom located at the point $i \in \mathbb{Z}^d$ where q is the *charge or coupling constant* and f is *single site potential*. Hence, our electron is exposed to a total potential

$$V(x) = \sum_{i\in\mathbb{Z}^d} qf(x-i).$$

Then, the potential V is periodic and the spectrum of the above operator H is well studied by using the Floquet-Bloch theory.

In contrast, when materials contain impurities or defects, the potential V is not periodic anymore. It depends on the possibly complicated configuration of impurities. Then, it is reasonable to consider V as a random quantity.

Following are some important examples of forms of random potentials:

Alloy-type model potential:

It models an unordered alloy, i.e., a mixture of several materials whose atoms are located at lattice positions. The type of atoms at each lattice point i is assumed to be random. Hence, the potential V is given by

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q_i(\omega)f(x-i).$$

where $q_i(\omega)$ are random variables which are usually assumed to be independent and identically distributed (i.i.d.).

Random displacement model:
By the random displacement model, we refer to a Schrödinger operator with the potential of the form:

$$V_\omega(x) = q \sum_{i \in \mathbb{Z}^d} f(x - i - \omega_i)$$

The above model appears in the case of pure materials where the atoms' positions randomly deviate from the ideal lattice randomly. Random variable ω_i in this case describes the deviation of the i^{th} atom from the lattice position i.
Poisson random model:
To model an amorphous material like glass or rubber, one assumes that the atoms of the material are located at random points in space. Then, the random potential is of the form:

$$V_\omega(x) = q \sum_{i \in \mathbb{Z}^d} f(x - \omega_i).$$

If we choose $\{\omega_i\}_{i \in Z^d}$ to be a Poisson point process in $\mathbb{R}^d$, the associated model is called the Poisson random model.
In statistical mechanics, we also study discrete random models i.e., the random models are defined on the space $l^2(\mathbb{Z}^d)$ instead of $L^2(\mathbb{R}^d)$. Following are two examples of discrete random models:
Discrete Anderson model:
The discrete Anderson model can be seen as a tight binding approximation of (2.1.1) in which electrons can hop from atom to atom and are subject to an external random potential modeling random environment:

$$H_\omega = -\Delta + V_\omega \tag{2.1.2}$$

where

- $-\Delta$ is the discrete Laplacian:

$$-\Delta u(n) = \sum_{|m-n|_1 = 1} u(m) \text{ for all } n \in \mathbb{Z}^d.$$

- V_ω is a multiplication operator:

$$(V_\omega u)(n) = V_\omega(n) u(n)$$

  where $\omega_n :== V_\omega(n)$ are i.i.d. random variables.

In dimension 1, the operator H_ω defined in (2.1.2) is nothing but a tridiagonal matrix of infinite length where the random variables ω_n only appear on the main diagonal of this matrix. That's why we also call H_ω the random model with diagonal disorder.
The following model is another example of discrete random operators but with off-diagonal disorder:

Lattice Hamiltonian with off-diagonal disorder:

For $u \in \mathbb{Z}^d$, one defines

$$(Mu)(x) = \sum_{\substack{y\in\mathbb{Z}^d:|y-x|_1=1 \\ e=\{x,y\}}} \gamma(e)(u(x)-u(y)) \tag{2.1.3}$$

for all $x \in \mathbb{Z}^d$, where $e = \{x, y\}$ is an un-oriented edges satisfying $|y-x|_1 = \sum_{k=1}^d |x_k - y_k| = 1$. Here $\{\gamma(e)\}$ are i.i.d. random variables. The model (2.1.3) appears in the description of waves (light, acoustic waves, etc) which propagate through a disordered, discrete medium (c.f. [Mia11] and references therein). We can see $\{\gamma(e)\}$ in this model as weights of bonds of the lattice $\mathbb{Z}^d$.
When $d = 1$, the random operator (2.1.3) can be rewritten in form of (1.1.1) introduced in Chapter 1. The model (2.1.3) in dimension 1 is our main object to study in Chapter 4. Throughout the thesis, whenever referring to the model (2.1.3) without other notifications, we will always assume following conditions on r.v.'s $\gamma(e)$:

(A.1) Random variables $\{\gamma(e)\}_{e\in\mathbb{Z}^d}$ have a common compactly supported bounded density ρ. Besides, the essential range of these r.v.'s is a finite interval

$$\mathrm{essRan}\gamma(e) = [\gamma^-, \gamma^+] \text{ with } 0 < \gamma^- < \gamma^+ < \infty$$

where the essential range essRan of a function $f : \Omega \mapsto \mathbb{R}$ is the set

$$\mathrm{essRan}f := \{x \in \mathbb{R} | \ \mathbb{P}\left(f^{-1}(x-\varepsilon, x+\varepsilon)\right) > 0 \text{ for all } \varepsilon > 0\}.$$

(A.2) The density ρ has a strictly positive lower bound in the sense that

$$\rho_- := \mathrm{ess} \inf_{s\in[\gamma^-,\gamma^+]} \rho(s) > 0.$$

2.2 Important quantities for random operators

In the present section, we only refer to discrete random operators but all definitions and results in this section could be applied to continuous settings as well.

2.2.1 Non random spectrum

Assume that H_ω is a discrete self-adjoint random operator on $l^2(\mathbb{Z}^d)$ depending on a collection of random variables ω. For each configuration of ω, we have a particular operator, hence, we have a family of self-adjoint operators. Each of them has its own spectrum.
Nevertheless, under some quite general conditions, the spectrum of a random operator is non-random i.e. H_ω, H_ω has the same spectrum ω almost surely. Following [Pas80, KS81, KM82], we shall precise conditions under which H_ω has the same spectrum ω almost surely.
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we call a measurable mapping $T : \Omega \to \Omega$ is a *measure preserving transformation* if $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$.
Let $\{T_k\}_{k\in\mathbb{Z}^d}$ be a family of measure preserving transformations, it is called *ergodic* iff

$$T_k^{-1}A = A \text{ for all } k \in \mathbb{Z}^d \implies \mathbb{P}(A) \in \{0, 1\}.$$

Now, assume that $\{H_\omega\}_{\omega\in\Omega}$ is a family of self-adjoint operators on $l^2(\mathbb{Z}^d)$. This family is *ergodic w.r.t. the additive group* $\mathbb{Z}^d$ if there exists a family of unitary operators $\{U_k\}_{k\in\mathbb{Z}^d}$ acting on $l^2(\mathbb{Z}^d)$ such that

$$H_{T_k\omega} = U_k^* H_\omega U_k.$$

In the case of the discrete Anderson model, $\{T_k\}_{k\in\mathbb{Z}^d}$ and $\{U_k\}_{k\in\mathbb{Z}^d}$ are nothing but shift and translation operators:

$$(T_k\omega)_i = \omega_{i-k}; \qquad (U_k\varphi)(i) = \varphi(i-k), \varphi \in l^2(\mathbb{Z}^d).$$

Following is the basic result of non-random spectra for random operators:

Theorem 2.2.1 ([Pas80, KS81, KM82])**.** *If H_ω is an ergodic family of self-adjoint operators, then there is a (closed, non random) subset Σ of $\mathbb{R}$, such that $\sigma(H_\omega) = \Sigma$ for $\mathbb{P}-a.s.$ ω.*
Moreover, there are sets $\Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp}$ such that $\sigma_{ac}(H_\omega) = \Sigma_{ac}, \sigma_{sc}(H_\omega) = \Sigma_{sc}, \sigma_{pp}(H_\omega) = \Sigma_{pp}$. Here $\sigma_{ac}, \sigma_{sc}, \sigma_{pp}$ stand for respectively the absolutely continuous, singular continuous and pure point spectrum of H_ω.

2.2.2 The integrated density of states

Consider a discrete ergodic random operator H_ω on $l^2(\mathbb{Z}^d)$. We can think of the discrete Anderson model as the typical example.
For any cube $\Lambda \in \mathbb{Z}^d$, we denote by $H_\omega(\Lambda)$ the operator H_ω restricted on Λ with periodic boundary condition. Then, $H_\omega(\Lambda)$ is nothing but a matrix acting on the finite dimensional

space $l^2(\Lambda) \equiv \mathbb{C}^\Lambda$. For any number $E \in \mathbb{R}$, let $N_\Lambda(E)$ be the number of eigenvalues of $H_\omega(\Lambda)$ less than or equal to E. The function $N_\Lambda(E)$ depends on the realization ω. By using the sub-additive ergodic theorem, following [Kir08], one has

$$N(E) = \lim_{|\Lambda| \to \infty} \frac{N_\Lambda(E)}{|\Lambda|}$$

exists $\omega-$a.s. for almost all $E \in \mathbb{R}$ and is $\omega-$independent.

One shows that $N(E)$ is independent of boundary conditions used to define it. We call $N(E)$ the *integrated density of states (IDS)* of H_ω. This quantity measures in some sense the number of states below a reference energy per unit volume. It is easy to see that $N(E)$ is nondecreasing and nonnegative. It thus can be viewed as the distribution function of a nonnegative measure on the real line. Besides, the support of this measure happens to equal the almost sure spectrum Σ of H_ω. Hence, $N(E)$ is constant outside Σ and the behavior of $N(E)$ as E approaches the boundary of Σ from inside is an interesting question. For example, let's consider the $d-$dimensional discrete Anderson model. Then, near the bottom E_0 of its a.s. spectrum, it is well known that $N(E)$ decay to 0 exponentially fast like $\exp[-c(E-E_0)^{-d/2}]$ with some positive constant c. This asymptotic of N is called the Lifschitz tail (c.f. [CL90] for a detailed discussion and especially Lifschitz's intuitive argument).

Another important question about IDS is its smoothness. It is a difficult one which attracted much attention but we do not intend to discuss that topic in the present thesis. We just want to mention that, as soon as a Wegner estimate (see Theorem 3.1.1 in Section 3.1) is known for H_ω, $N(E)$ is defined everywhere in $\mathbb{R}$ and absolutely continuous w.r.t. Lebesgue measure with a bounded derivative $\nu(E)$ called the *density of states (DoS)* of H_ω. Interested readers can find much further discussions and results in the smoothness of IDS in e.g. [His08, Ves08, Kir08, CL90].

Finally, the definition of IDS and DoS for continuous random operators is similar provided that, for $\Lambda \subset \mathbb{R}^d$, $H_\omega(\Lambda)$ is lower semibounded and has a discrete spectrum. Note that, in the continuous case, the problem of boundary conditions is more complicated.

CHAPTER 3

Local level statistics in localized regime

Contents

3.1 The Wegner estimate

The term *Wegner estimate* refers to an upper bound on the probability that the finite-volume random operator $H_\omega(\Lambda)$ has an eigenvalue in a given interval I. A *good* Wegner estimate is one for which the upper bound depends linearly on the volume $|\Lambda|$ and on the length side of the interval:

$$\mathbb{P}\left(\sigma(H_\omega\Lambda) \cap I \neq \emptyset\right) \leq C|I||\Lambda|.$$

Such an estimate is a key ingredient to prove the Anderson localization as well as to study the smoothness of IDS and the spectral statistics of random models.
Up to now, the *good* Wegner-type estimates have been obtained for many random Schrödinger models both discrete and continuous ones under rather general conditions on the single-site potential and on the randomness (c.f. [His08, Ves08] for detailed surveys).
As referred in Chapter 2, the spectral statistics for the lattice Hamiltonian with off-diagonal (2.1.3) in dimension 1 will be studied in Chapter 4 and a Wegner-type estimate for (2.1.3) will be used. For that reason, we would like to state here the Wegner estimate for (2.1.3) in the present chapter:

Theorem 3.1.1. *[Wegner estimate, [Mia11, Theorem 2.1]] Pick $\Lambda \in \mathbb{Z}^d$ a large cube and $E > 0$. Let $I_\varepsilon = [E-\varepsilon, E+\varepsilon]$ where ε is a small number between 0 and E. Denote by M_Λ the operator M in (2.1.3) restricted to Λ with the periodic boundary conditions.*
Then, we have

$$\mathbb{P}\left(\sigma(M_\Lambda) \cap I_\varepsilon \neq \emptyset\right) \leq \frac{2d\|s\rho(s)\|_\infty}{E-\varepsilon}\varepsilon|\Lambda|$$

for all $\Lambda \subset \mathbb{Z}$ and $0 < \varepsilon < E$.

3.2 Minami estimate

Minami estimate (M) is one the most crucial ingredients for studying spectral statistics and an essential ingredient in our proof of decorrelation estimates (see Chapter 4). (M) is well known for many discrete random models under assumptions of the regularity of randomness. For examples, this estimate holds for the discrete Anderson model everywhere in the almost sure spectrum and in any dimension (c.f. [Min96, GV07, BHS07, TV14]).
For the continuous models, Minami-type estimates have not been settled yet except for random models in dimension 1. Actually, the authors in [Klo14] showed that, for some 1D random Schrödinger operators, (W) implies (M) in the localized regime.

Minami [Min96], for the first time, introduced an upper bound for the probability of having two or more eigenvalues in a small interval of energies as a key to obtain the Poisson statistic for eigenvalues of multi-dimensional discrete Anderson tight-binding model. That bound is now widely known under the name Minami estimate:

$$\mathbb{P}(\#\{\sigma(H_\omega(\Lambda)) \cap J\} \geqslant 2) \leqslant C(|J||\Lambda|)^2$$

where H_ω here denote the multi-dimensional Anderson tight-binding model.
It is worth mentioning that Minami estimate and localization properties imply the simplicity of the spectrum.
Minami's original proof of this estimate relies on finding a good upper bound on the average of the determinant whose entries are matrix elements of the imaginary part of the resolvent:

Lemma 3.2.1. *[Min96, Lemma 2] For any $z \in \mathbb{C}_+$, $D \in \mathbb{Z}^d$, and $x, y \in D$ with $x \neq y$, one has*

$$\mathbb{E}\left[\det\begin{pmatrix} ImG^D(z;x,x) & ImG^D(z;x,y) \\ ImG^D(z;y,x) & ImG^D(z;y,y)\end{pmatrix}\right] \leqslant \pi^2\|\rho\|_\infty^2$$

Recently, a simpler and more transparent proof of (M) for the discrete Anderson model is given by Combes, Germinet and Klein [CGK09]. Their proof bases on an averaging spectral projections and rank one perturbation and requires not too much of the regularity of the random variables $\{\omega_n\}_{n\in\mathbb{Z}^d}$. Moreover, that strategy can be adapted to other (discrete) random models such as the lattice Hamiltonian with off-diagonal disorder (2.1.3):

Theorem 3.2.2. *[Mia11, Theorem 3.1] Pick $\Lambda = [-L, L]^d \subset \mathbb{Z}^d$. Let M_Λ be the lattice Hamiltonian with off-diagonal defined in (2.1.3) restricted on Λ with periodic boundary conditions. We assume the conditions (A1) and (A2) for r.v.'s $\gamma(e)$ of (2.1.3).*
Then, there exists $C > 0$ such that, for all intervals $J = [a, b] \subset (0, +\infty)$, one has

$$\mathbb{P}\left(tr\mathbb{1}_I(M_\Lambda) \geqslant 2\right) \leqslant \beta_0\|\rho\|_\infty\|s\rho(s)\|_\infty(|J||\Lambda|)^2/a^2.$$

3.3 The localized regime

We would like to begin this section with a phenomenal discovery in condensed-matter physics by American physicist Philip Warren Anderson. For the first time, in the 1950s, he suggested the occurrence of localized electronic states in disordered systems provided that the degree of randomness of the impurities or defects is sufficiently large. Such a phenomenon is now widely known under the name *Anderson localization*. It is of importance in different contexts, e.g., it plays a role in the conductive properties of semi conductor materials, in the quantization of Hall conductance or in some subjects of optical crystals. Anderson localization mathematically means that the spectrum of random operators is almost surely *pure point* (consists of a dense collection of eigenvalues) corresponding to exponentially decaying eigenfunctions. We define the *localized regime (or a region of localization)* as the region in the almost sur spectrum where Anderson localization is exhibited. Interested readers can find physical reviews of Anderson localization in [And58, Tho86]. For mathematical proofs of Anderson localization with "regular" random potentials, see [GMP77, FS83, ASFH01, GK01]. For those of Anderson localization with singular random potentials, see [CKM87, DSS02, BK05, GK13].

For random operators in dimension 1, the knowledge of mathematicians about localized regime is quite satisfactory. It is well known that Anderson localization occurs everywhere within the almost sure spectrum of 1D random operators. In the other words, one dimensional disordered systems ("thin wire with impurities") should have low or even vanishing conductivity.
However, in higher dimensions, much less is known about localized regime. In these cases, Anderson localization is only proven at low energies, high disorder or at the spectral edges:

- For the discrete Anderson model, let S_- and S_+ be the infimum and supremum of Σ. Then, for some $S_- < s_- \leqslant s_+ < S_+$, the intervals $I_- = [S_-, s_-)$ and $I_+ = (s_+, S_+]$ are contained in the localized regime H_ω. Besides, one can pick $I := I_- \cup I_+ = \Sigma$ in the high disordered case (c.f. [Kir08]).
- For random wave operator (2.1.3), Anderson localization is proved in the cases of high frequencies/energies ([Far87, Far91]) and frequencies/energies near band edges (c.f.[FK94] for discrete case and [FK96] for continuous case).

We have two powerful methods to prove the localization for multidimensional random operators, the multiscale analysis and the fractional moment. The former is a technique initially introduced by Fröhlich and Spencer [FS83], and simplified by von Dreifus and Klein [vDK89], and strengthened in various directions by Germinet, Klein [GK01]. We recommend [Kir08] to interested readers for a very detailed survey on this topic.
The fractional moment method was first developed by Aizenman and Molchanov [AM93] for discrete Schrödinger operators . It was then extended to the continuous setting by Aizenman, Elgart, Naboko, Schenker and Stolz in [AEN+06].

Throughout the present thesis, we use the following definition of the localized regime:

Proposition 3.3.1. *[Klo11] Let I be the region of Σ where the finite volume fractional moment criteria of [ASFH01] for $H_\omega(\Lambda)$ are verified for Λ sufficiently large.*
Then,
(Loc): *There exists $\nu > 0$ such that, for any $p > 0$, there exists $q > 0$ and $L_0 > 0$ such that, for $L \geqslant L_0$, with probability larger than $1 - L^{-p}$, if*

1. *$\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda_L)$ associated to an energy $E_{n,\omega} \in I$,*
2. *$x_{n,\omega} \in \Lambda_L$ is a maximum of $x \mapsto |\varphi_{n,\omega}(x)|$ in Λ_L,*

then, for $x \in \Lambda_L$, one has

$$|\varphi_{n,\omega}(x)| \leqslant L^q e^{-\nu|x - x_{n,\omega}|}.$$

The point $x_{n,\omega}$ is called a localization center for $\varphi_{n,\omega}$ or $E_{n,\omega}$.

3.4 Poisson process (c.f. [DVJ08])

Let $M(\mathcal{R})$ be the space of all non-negative Radon measures on $\mathbb{R}$. On this space, we define a so-called vague topology as follows: We say that $\{\mu_n\}_{n\geq 1}$ in $M(\mathcal{R})$ converges to $m \in M(\mathcal{R})$ vaguely iff

$$\lim_{n\to+\infty} \int f(x)\mu_n(dx) = \int f(x)\mu(dx)$$

for all f belonging to $C_c^+(\mathbb{R})$, the space of non-negative continuous functions with compact support.
Now, we denote by $M_p(\mathbb{R})$ the sub-space of $M(\mathcal{R})$ consisting of all integer valued Radon measures on $\mathbb{R}$. Any measure $\xi \in M_p(\mathbb{R})$ can be written as

$$\xi(dx) = \sum_j \delta_{\xi_j}(dx)$$

with a sequence ξ_j having no finite accumulation points.

Definition 3.4.1. *A point process is a random variable $\xi := \xi^\omega$ which takes values in $M_p(\mathbb{R})$. In addition, the intensity measure $\mu(dx)$ of ξ^ω is defined as*

$$\mu(A) = \mathbb{E}[\xi^\omega(A)] \text{ for any Borel set } A\ .$$

Consider a sequence $\{\xi_n\}$ of point processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Following are equivalent conditions for the convergence of $\{\xi_n\}$ to a point process ξ which may be defined on another probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ (c.f. [DVJ08]):

1. For any bounded continuous function Φ on $M_p(\mathbb{R})$ one has
$$\lim_{n\to\infty} \int \Phi(\xi_n)\mathbb{P}(d\omega) = \int \Phi(\xi)\overline{\mathbb{P}}(d\omega).$$
2. For any $f \in C_c^+(\mathbb{R})$, set $\xi_n(f) := \int f(x)\xi_n(dx)$, one has
$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[e^{-\xi_n(f)}] = \mathbb{E}_{\overline{\mathbb{P}}}[e^{-\xi(f)}].$$
3. For any $l \geq 1, \{k_j\}_{1\leq j\leq l} \in \mathbb{N}^*$ and disjoint intervals $\{I_j\}_{1\leq j\leq l}$ such that $\overline{\mathbb{P}}(\xi(\partial I_j) > 0) = 0$, one has
$$\lim_{n\to\infty} \mathbb{P}(\xi_n(I_j) = k_j, j = 1, \ldots, l) = \overline{\mathbb{P}}(\xi_n(I_j) = k_j, j = 1, \ldots, l).$$

Now we define a Poisson point process (Poisson process for short).

Definition 3.4.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A point process ξ defined on this probability space is called Poisson point process with intensity measure ν if the following two statements hold:*

- *For any bounded Borel set $S \subset \mathbb{R}$, the random variable $\xi(S)$ has a Poisson distribution with parameter $\nu(S)$ i.e.,*
$$\mathbb{P}(\xi(S) = k) = \frac{\nu(S)^k}{k!}e^{-\nu(S)} \text{ for all } k \geqslant 0.$$
- *If bounded Borel sets $S_1, \ldots, S_k$ in $\mathbb{R}$ are mutually disjoint, then the random variables $\xi(S_1), \ldots, \xi(S_k)$ are independent.*

In particular, if the measure ν happens to be the Lebesgue measure on $\mathbb{R}$, the point process ξ is called the Poisson process with the intensity 1.

3.5 Local level statistics in the localized regime

Let's consider an arbitrary $\mathbb{Z}^d-$ergodic, discrete random operator H_ω on $l^2(\mathbb{Z}^d)$. The very usual way to study various spectral statistics of this operator is to use the finite-volume approximation operators. We restrict H_ω on a cube Λ, say $[-L, L]^d$ with some boundary conditions to get finite-volume operator denoted by $H_\omega(\Lambda)$, a symmetric matrix.

Hence, $H_\omega(\Lambda)$ has only real eigenvalues and we study the properties of its eigenvalues in the limit $|\Lambda| \to +\infty$.
Assume that the integrated density of states $N(E)$ and the density of states $\nu(E)$ exist for the random operator H_ω. Then, for any interval $I = [a, b]$, for Λ sufficiently large, we have

$$\#\{\text{e.v. of } H_\omega(\Lambda) \in I\} \asymp N(I)|\Lambda| \text{ where } N(I) = N(b) - N(a). \tag{3.5.1}$$

(3.5.1) yield that the *mean spacing*, the average distance between eigenvalues of $H_\omega(\Lambda)$, in I is of order $|\Lambda|^{-1}$ as $|\Lambda|$ large.
Now, pick a fixed, arbitrary energy E in Σ, the almost sure spectrum of H_ω such that $\nu(E) > 0$. By the *local level statistics* near E, we mean the following point process

$$\Xi(\xi, E, \omega, \Lambda) = \sum_{n=1}^{|\Lambda|} \delta_{\xi_n}(E, \omega, \Lambda)(\xi) \tag{3.5.2}$$

where

$$\xi_n(E, \omega, \Lambda) := |\Lambda|\nu(E)(E_n(\omega, \Lambda) - E) \text{ are rescaled eigenvalues.} \tag{3.5.3}$$

For any Borelian set $A \subset \mathbb{R}$, $\Xi(A, E, \omega, \Lambda)$ is nothing but the number of ξ_n which belongs to A. On the other hand, we observe that $\xi_n \in A$ iff $E_n \in E + (\nu(E)|\Lambda|)^{-1}A$. Hence, if $|\Lambda|$ is large enough and $\nu(E) > 0$, only eigenvalues E_n close to the fixed energy E are taken into account in the point process (3.5.2). That's why we call $\Xi(\xi, E, \omega, \Lambda)$ the *local* level statistics near E.
Let's take a look at the definition of rescaled eigenvalues $\xi_n(E, \omega, \Lambda)$. We knew that, typically, consecutive eigenvalues of $H_\omega(\Lambda)$ as $|\Lambda|$ large are close to each other (the mean spacing is in magnitude of $|\Lambda|^{-1}$). Hence, by multiplying eigenvalues by $|\Lambda|$ as in the definition of $\xi_n(E, \omega, \Lambda)$, we obtain a rescaled spectrum where the typical distance between two rescaled eigenvalues is constant in magnitude. Such a rescaling procedure is very natural and obviously makes the study of spectral statistics easier. Finally, there is a "renormalization factor" $\nu(E) > 0$ appearing in the definition of $\xi_n(E, \omega, \Lambda)$ just because we would like that, in the conclusion of Theorem 3.5.1, the limit of the point process $\Xi(\xi, E, \omega, \Lambda)$ is a Poisson process with the intensity 1 instead of $\nu(E)$.
Note that, to study the local level statistics near an energy $E \in \Sigma_{\mathbb{Z}}$, we can also consider the point process Ξ defined in (3.5.2) with $\xi_n(E, \omega, \Lambda) = |\Lambda|(N(E_n) - N(E))$. The advantage when considering this point process is that we do not need an assumption on the density ν anymore.
Following is one of the most striking results of local level statistics in the localized regime

for discrete random operators. Roughly speaking, in the subregion of the localized regime where the density of states ν does not vanish, eigenvalues of $H_\omega(\Lambda)$ are distributed in Poissonian way locally in the limit $|\Lambda| \to +\infty$.

Theorem 3.5.1. *[Mol81, Min96, GK14] Consider a $\mathbb{Z}^d-$ergodic random operator H_ω on $l^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$ satisfying the independence at a distance (IAD):*
(IAD): *There exists $R_0 > 0$ such that for $dis(\Lambda, \Lambda') > R_0$, the random Hamiltonians $H_\omega(\Lambda)$ and $H_\omega(\Lambda')$ are independent.*
*Let E be a fixed energy in the localized regime of H_ω (where **(Loc)** in Proposition 3.3.1 holds) such that $\nu(E) > 0$. In addition, assume that Wegner estimate and Minami estimate hold in a neighborhood of E.*
Then, when $|\Lambda| \to +\infty$, the point process $\Xi(\xi, E, \omega, \Lambda)$ converges weakly to a Poisson point process with the intensity 1 i.e., for $(U_j)_{1 \leqslant j \leqslant J}$, $U_j \subset \mathbb{R}$ bounded measurable and $U_{j'} \cap U_j = \emptyset$ if $j \neq j'$ and $(k_j)_{1 \leqslant j \leqslant J} \in \mathbb{N}^J$, we have

$$\lim_{|\Lambda| \to +\infty} \left| \mathbb{P}\left(\begin{Bmatrix} \#\{j; \xi_j(E,\omega,\Lambda) \in U_1\} & = k_1 \\ \vdots & \vdots \\ \#\{j; \xi_j(E,\omega,\Lambda) \in U_j\} & = k_J \end{Bmatrix} \right) - \prod_{j=1}^{J} \frac{|U_j|^{k_j}}{k_j!} e^{-|U_j|} \right| = 0.$$

This kind of result was first proved in 1981 by Molchanov [Mol81] for some one-dimensional continuous random Schrödinger operator. His result was extended fifteen years later by Minami [Min96] for the discrete Anderson model and Minami's proof is independent of the dimension of phase space.

Recently, the results on local level statistics have been studied intensively and extended in various directions by Germinet and Klopp [GK14]. In [GK14], the authors obtained a general version of Poisson convergence which can be resumed as follows:
*For any $\mathbb{Z}^d-$ergodic random operator H_ω on $l^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$, the Poisson convergence holds true whenever we have **(IAD)**, **(Loc)**, Wegner and Minami estimates.*
Moreover, the authors in [GK14] weakened the condition $\nu(E)$ in Theorem 3.5.1 and derived a *uniform* Poisson convergence over small intervals of energy instead of some fixed energy E.

CHAPTER 4

Decorrelation estimates in dimension 1

Contents

4.1 Definitions and statement of results

In the present chapter, we would like to study local level statistics of the lattice Hamiltonian with off-diagonal disorder (2.1.3) in dimension 1. In dimension 1, (2.1.3) can be rewritten in the following form: for $u = \{u(n)\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$, set

$$(H_\omega u)(n) = \omega_n(u(n) - u(n+1)) - \omega_{n-1}(u(n-1) - u(n)). \tag{4.1.1}$$

Throughout the present chapter, we assume that $\omega := \{\omega_n\}_{n\in\mathbb{Z}}$ are non-negative i.i.d. random variables (r.v.'s for short) with a bounded, compactly supported density ρ.
In addition, from Section 4.1 to Section 4.2, we assume moreover that $\omega_n \in [\alpha_0, \beta_0]$ for all $n \in \mathbb{Z}$ where $\beta_0 > \alpha_0 > 0$. In Section 4.3, we will comment on relaxing the hypothesis of the lower bound of r.v.'s ω.
It is known that (see [Mia11]):

- the operator H_ω admits an almost sure spectrum $\Sigma := [0, 4\beta_0]$.
- H_ω has an integrated density of states defined as follows:
  $\omega-$a.s., the following limit exists and is ω independent:
  $$N(E) := \lim_{|\Lambda|\to+\infty} \frac{\#\{\text{e.v. of } H_\omega(\Lambda) \text{ less than E}\}}{|\Lambda|} \text{ for a.e. } E \tag{4.1.2}$$
  where $H_\omega(\Lambda)$ is the operator H_ω restricted on the interval $\Lambda \subset \mathbb{Z}$ with some boundary condition.
  As a direct consequence of the Wegner estimate (see Theorem 3.1.1 in Section 3.1), $N(E)$ is defined everywhere in $\mathbb{R}$ and absolutely continuous w.r.t. Lebesgue measure with a bounded derivative $\nu(E)$ called the density of states of H_ω.

In the present chapter, we follow a usual way to study various statistics related to random operators. We restrict the operator H_ω on some interval $\Lambda \subset \mathbb{Z}$ of finite length with some boundary condition and obtain a finite-volume operator which is denoted by $H_\omega(\Lambda)$. Then, we study diverse statistics for this operator in the limit when $|\Lambda|$ goes to infinity.
Throughout this chapter, the boundary condition to define $H_\omega(\Lambda)$ is always the periodic boundary condition. For example, if $\Lambda = [1, N]$, the operator $H_\omega(\Lambda)$ is a symmetric $N \times N$ matrix of the following form:

$$\begin{pmatrix} \omega_N + \omega_1 & -\omega_1 & 0 & \dots & 0 & -\omega_N \\ -\omega_1 & \omega_1 + \omega_2 & -\omega_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \omega_{N-2} + \omega_{N-1} & -\omega_{N-1} \\ -\omega_N & 0 & 0 & \dots & -\omega_{N-1} & \omega_{N-1} + \omega_N \end{pmatrix}.$$

For $L \in \mathbb{N}$, let $\Lambda = \Lambda_L := [-L, L]$ be a large interval in $\mathbb{Z}$ and $|\Lambda| := (2L+1)$ be its cardinality.
We will denote the eigenvalues of $H_\omega(\Lambda)$ ordered increasingly and repeated according to multiplicity by $E_1(\omega,\Lambda) \leqslant E_2(\omega,\Lambda) \leqslant \cdots \leqslant E_{|\Lambda|}(\omega,\Lambda)$.
Let I be the localized regime (the region of localization) in Σ where the finite-volume fractional-moment criteria for localization are satisfied for the finite-volume operators $H_\omega(\Lambda)$ when $|\Lambda|$ is large enough (see Proposition 3.3.1 in Section 3.3 and [ASFH01] for more details).
Pick E an energy in I with $\nu(E) > 0$ and define the *local level statistics* near E as in Chapter 3:

$$\Xi(\xi, E, \omega, \Lambda) = \sum_{n=1}^{|\Lambda|} \delta_{\xi_n}(E, \omega, \Lambda)(\xi) \tag{4.1.3}$$

where

$$\xi_n(E, \omega, \Lambda) = |\Lambda|\nu(E)(E_n(\omega,\Lambda) - E). \tag{4.1.4}$$

From Theorem 3.1.1 and Theorem 3.2.2 in Chapter 3, Wegner estimate and Minami estimate for the model (4.1.1) do not hold at 0, the bottom of its almost sure spectrum Σ. Hence, we have to restrict ourselves to the study of the local level statistics $\Xi(\xi, E, \omega, \Lambda)$ with $E > 0$ in the localized regime.
For the model (4.1.1), it is known that the weak limit of the above point process is a Poisson point process:

Theorem 4.1.1. *[Mia11] Assume that E is a positive energy in I with $\nu(E) > 0$. Then, when $|\Lambda| \to +\infty$, the point process $\Xi(\xi, E, \omega, \Lambda)$ converges weakly to a Poisson point process with the intensity 1 i.e., for $(U_j)_{1\leqslant j\leqslant J}$, $U_j \subset \mathbb{R}$ bounded measurable and $U_{j'} \cap U_j = \emptyset$ if $j \neq j'$ and $(k_j)_{1\leqslant j \leqslant J} \in \mathbb{N}^J$, we have*

$$\lim_{|\Lambda|\to+\infty} \left| \mathbb{P}\left(\left\{ \begin{array}{ll} \#\{j; \xi_j(E,\omega,\Lambda) \in U_1\} & = k_1 \\ \vdots & \vdots \\ \#\{j; \xi_j(E,\omega,\Lambda) \in U_j\} & = k_J \end{array} \right\} \right) - \prod_{j=1}^{J} \frac{|U_j|^{k_j}}{k_j!} e^{-|U_j|} \right| = 0.$$

Recently, for the 1D discrete Anderson model, Klopp [Klo11] showed moreover that if we pick two fixed, distinct energies E and E' in the localized regime, the two corresponding point processes $\Xi(\xi, E, \omega, \Lambda)$ and $\Xi(\xi, E', \omega, \Lambda)$ converge weakly, respectively to two independent Poisson point processes. In other words, the limits of $\Xi(\xi, E, \omega, \Lambda)$ and $\Xi(\xi, E', \omega, \Lambda)$ are stochastically independent.

It is known that the above statement holds true if one can prove a so-called decorrelation estimate.
That is exactly what we want to carry out here for the 1D discrete lattice Hamiltonian with off-diagonal disorder (4.1.1). Our decorrelation estimate is the following:

Theorem 4.1.2. *Let E, E' be two positive, distinct energies in the localized regime. Pick $\beta \in (1/2, 1)$ and $\alpha \in (0,1)$. Then, for any $c > 0$, there exists $C > 0$ such that, for L large enough and $cL^\alpha \leqslant l \leqslant L^\alpha/c$, one has*

$$\mathbb{P}\left(\begin{Bmatrix} \sigma(H_\omega(\Lambda_l)) \cap (E + L^{-1}(-1,1)) \neq \emptyset \\ \sigma(H_\omega(\Lambda_l)) \cap (E' + L^{-1}(-1,1)) \neq \emptyset \end{Bmatrix}\right) = o\left(\frac{l}{L}\right).$$

Thanks to Theorem 4.1.2, we can proceed as in Section 3 of [Klo11] to obtain the asymptotic independence of the weak limits of $\Xi(\xi, E, \omega, \Lambda)$ and $\Xi(\xi, E', \omega, \Lambda)$ with $E, E' > 0$ for the model (4.1.1):

Theorem 4.1.3. *Pick two positive, distinct energies E and E' in the localized regime such that $\nu(E) > 0$ and $\nu(E') > 0$.*
When $|\Lambda| \to +\infty$, the point processes $\Xi(\xi, E, \omega, \Lambda)$, and $\Xi(\xi, E', \omega, \Lambda)$ converge weakly respectively to two independent Poisson processes on $\mathbb{R}$ with intensity the Lebesgue measure. That is, for $(U_j)_{1\leqslant j\leqslant J}$, $U_j \subset \mathbb{R}$ bounded measurable and $U_{j'} \cap U_j = \emptyset$ if $j \neq j'$ and $(k_j)_{1\leqslant j\leqslant J} \in \mathbb{N}^J$ and $(U'_j)_{1\leqslant j\leqslant J'}$, $U'_j \subset \mathbb{R}$ bounded measurable and $U'_{j'} \cap U'_j = \emptyset$ if $j \neq j'$ and $(k'_j)_{1\leqslant j\leqslant J'} \in \mathbb{N}^{J'}$, we have

$$\mathbb{P}\left(\left\{\begin{array}{ll} \#\{j; \xi_j(E,\omega,\Lambda) \in U_1\} & = k_1 \\ \vdots & \vdots \\ \#\{j; \xi_j(E,\omega,\Lambda) \in U_J\} & = k_J \\ \#\{j; \xi_j(E',\omega,\Lambda) \in U'_1\} & = k'_1 \\ \vdots & \vdots \\ \#\{j; \xi_j(E',\omega,\Lambda) \in U'_J\} & = k'_J \end{array}\right\}\right) \to \prod_{j=1}^{J} \frac{|U_j|^{k_j}}{k_j!} e^{-|U_j|} \prod_{i=1}^{J'} \frac{|U'_i|^{k'_i}}{k'_i!} e^{-|U'_i|} \tag{4.1.5}$$

as $|\Lambda| \to +\infty$.

Moreover, in Section 4.4, we will generalize Theorem 4.1.3 by considering not only two but any fixed number of distinct energies.
To prove Theorem 4.1.2 for the model (4.1.1), we follow the strategy introduced in [Klo11]. The key point of the proof of decorrelation estimates for the 1D discrete Anderson model

in [Klo11] is to derive that the gradient with respect to ω of two eigenvalues $E(\omega)$ and $E'(\omega)$ near two distinct energies E and E' are not co-linear with a good probability. In the discrete Anderson case, this statement will hold true if the gradients of $E(\omega)$ and $E'(\omega)$ are distinct. In deed, the gradients of $E(\omega)$ and $E'(\omega)$ have non-negative components and their l^1-norm are always equal to 1. So, if they are co-linear, they should be the same.
Unfortunately, the l^1-norm of the gradient of an eigenvalue of our finite-volume operator $H_\omega(\Lambda)$ is not a constant w.r.t. ω anymore (it is even not bounded from below by a positive constant uniformly w.r.t. ω). Moreover, to prove the above key point for the discrete Anderson model, [Klo11] exploits the diagonal structure of the potential which can not be used for the present case. So, a significant modification in the proof is needed to obtain Theorem 4.1.2. This approach is contained in Lemma 4.2.3. Besides, in Section 4.5, we show that that new approach can be adapted to the discrete Anderson model in dimension 1 as well.
In addition, Theorem 1.12 in [GK11b] implies directly the following result for the model (4.1.1):

Theorem 4.1.4. *[GK11b, Theorem 1.12] Pick $0 < E_0 \in I$ such that the density of states ν is continuous and positive at E_0.*
Consider two sequences of positive energies, say $(E_\Lambda)_\Lambda$, $(E'_\Lambda)_\Lambda$ such that

1. $E_\Lambda \xrightarrow[\Lambda\to\mathbb{Z}^d]{} E_0$ *and* $E'_\Lambda \xrightarrow[\Lambda\to\mathbb{Z}^d]{} E_0$,
2. $|\Lambda||N(E_\Lambda) - N(E'_\Lambda)| \xrightarrow[\Lambda\to\mathbb{Z}^d]{} +\infty$.

Then, the point processes $\Xi(\xi, E_\Lambda, \omega, \Lambda)$ and $\Xi(\xi, E'_\Lambda, \omega, \Lambda)$ converges weakly respectively to two independent Poisson point processes in $\mathbb{R}$ with intensity the Lebesgue measure.

In Theorem 4.1.4, instead of fixing two distinct energies E and E', one considers two sequences of positive energies $\{E_\Lambda\}$, $\{E'_\Lambda\}$ which tend to each other as $|\Lambda| \to \infty$. In addition, one assumes that, the distance between two points processes $\Xi(\xi, E_\Lambda, \omega, \Lambda)$ and $\Xi(\xi, E'_\Lambda, \omega, \Lambda)$ goes to infinity as $|\Lambda| \to \infty$. Then, the asymptotic independence of two point processes associated to E_Λ and E'_Λ is obtained.
Besides, it is known that the existence of an integrated density of states defined as in (4.1.2) implies that the average distance (mean spacing) between eigenlevels is of order $|\Lambda|^{-1}$.
Thus, according to Theorem 4.1.4, in the localized regime, eigenvalues separated by a distance that is asymptotically infinite with respect to the mean spacing between eigenlevels behave like independent random variables. In other words, there are no interactions between distinct eigenvalues, except at a very short distance.

Notation: In the present chapter, we use Dirac's notations: If φ is a vector in a Hilbert space $\mathcal{H}$, we denote by $|\varphi\rangle\langle\varphi| = \langle\varphi,\cdot\rangle_{\mathcal{H}}\varphi$ the projection operator on φ. Besides, throughout the present chapter, the symbol $\|\cdot\|$ stands for the l^2-norm $\|\cdot\|_2$ in some finite dimensional Hilbert space.

4.2 Proof of Theorem 4.1.2

In the present section, we follow the strategy introduced in [Klo11] to prove the decorrelation estimate in Theorem 4.1.2.

Pick two distinct, positive energies E, E' in I (the localized regime). Let $J_L = E + L^{-1}[-1,1]$ and $J'_L = E' + L^{-1}[-1,1]$ with L large. We would like to begin this section by proving some elementary properties of eigenvalues of $H_\omega(\Lambda)$ with an arbitrary interval $\Lambda \in \mathbb{Z}$.

Lemma 4.2.1. *Suppose that $\omega \mapsto E(\omega)$ is the only eigenvalue of $H_\omega(\Lambda)$ in J_L. Then*

1. *$E(\omega)$ is simple and $\omega \mapsto E(\omega)$ is real analytic. Moreover, let $\omega \mapsto \varphi(\omega)$ denote the real-valued, normalized eigenvector associated to $E(\omega)$, it is also real analytic in ω.*
2. *$\|\nabla_\omega E(\omega)\|_1 = 2\sum_{\gamma\in\Lambda}\|\Pi_\gamma\varphi\|^2$ where $\Pi_\gamma = \dfrac{1}{2}|\delta_\gamma - \delta_{\gamma+1}\rangle\langle\delta_\gamma - \delta_{\gamma+1}|$ is a projection in $l^2(\Lambda)$. Besides, we have $E(\omega) \in [0, 4\beta_0]$.*
3. *$Hess_\omega E(\omega) = (h_{\gamma,\beta})_{\gamma,\beta}$ where*
   - $h_{\gamma,\beta} := -4\Big\langle \big(H_\omega(\Lambda) - E(\omega)\big)^{-1}\psi_\gamma, \psi_\beta\Big\rangle$,
   - $\psi_\gamma := \langle\Pi_\gamma\varphi,\varphi\rangle\varphi - \Pi_\gamma\varphi = -\Pi_{\langle\varphi\rangle^\perp}(\Pi_\gamma\varphi)$ *where $\Pi_{\langle\varphi\rangle^\perp}$ is the orthogonal projection on $\langle\varphi\rangle^\perp$.*

Proof of Lemma 4.2.1. (1) is true from the standard perturbation theory (c.f. [Kat95]). Now we will prove (2). Starting from the eigenequation

$$H_\omega(\Lambda)\varphi = E(\omega)\varphi, \tag{4.2.1}$$

we have, for all $\gamma \in \Lambda$,

$$\begin{aligned}
\partial_{\omega_\gamma}E(\omega) &= \langle\partial_{\omega_\gamma}(H_\omega(\Lambda)\varphi),\varphi\rangle + \langle H_\omega(\Lambda)\varphi, \partial_{\omega_\gamma}\varphi\rangle \\
&= \langle\partial_{\omega_\gamma}(H_\omega(\Lambda))\varphi,\varphi\rangle + \langle H_\omega(\Lambda)\partial_{\omega_\gamma}\varphi,\varphi\rangle + \langle H_\omega(\Lambda)\varphi,\partial_{\omega_\gamma}\varphi\rangle \\
&= \langle\partial_{\omega_\gamma}(H_\omega(\Lambda))\varphi,\varphi\rangle + \langle\partial_{\omega_\gamma}\varphi, H_\omega(\Lambda)\varphi\rangle + \langle H_\omega(\Lambda)\varphi,\partial_{\omega_\gamma}\varphi\rangle \\
&= \langle\partial_{\omega_\gamma}(H_\omega(\Lambda))\varphi,\varphi\rangle + E(\omega)\big(\langle\partial_{\omega_\gamma}\varphi,\varphi\rangle + \langle\varphi,\partial_{\omega_\gamma}\varphi\rangle\big)
\end{aligned}$$

where the last two equalities come from the symmetry of $H_\omega(\Lambda)$ and (4.2.1).
Noting that

$$\langle \partial_{\omega_\gamma}\varphi, \varphi\rangle + \langle \varphi, \partial_{\omega_\gamma}\varphi\rangle = 2\partial_{\omega_\gamma}\|\varphi\|^2 = 0.$$

Hence,

$$\partial_{\omega_\gamma}E(\omega) = \langle \partial_{\omega_\gamma}(H_\omega(\Lambda))\varphi, \varphi\rangle = 2\langle \Pi_\gamma\varphi, \varphi\rangle. \tag{4.2.2}$$

On the other hand, it is easy to check that $\Pi_\gamma = \Pi_\gamma^* = \Pi_\gamma^2$. Hence, Π_γ is an orthogonal projection and $\partial_{\omega_\gamma}E(\omega) = 2\|\Pi_\gamma\varphi\|^2$.
Thanks to (4.2.1) and (4.2.2), we have the following important equality:

$$\sum_{\gamma\in\Lambda}\omega_\gamma\partial_{\omega_\gamma}E(\omega) = 2\sum_{\gamma\in\Lambda}\omega_\gamma\langle \Pi_\gamma\varphi, \varphi\rangle = E(\omega) \tag{4.2.3}$$

which characterize the form of our operator.
From (4.2.3) and $\|\varphi\| = 1$, we infer that

$$0 \le E(\omega) = \sum_{\gamma\in\Lambda}\omega_\gamma(\varphi(\gamma) - \varphi(\gamma+1))^2 \le 4\beta_0. \tag{4.2.4}$$

Finally, we give a proof for (3). By differentiating both sides of (4.2.2) w.r.t. ω_γ, we have

$$\begin{aligned}\partial^2_{\omega_\gamma}E(\omega) &= 2\langle \partial_{\omega_\gamma}(\Pi_\gamma\varphi), \varphi\rangle + 2\langle \Pi_\gamma\varphi, \partial_{\omega_\gamma}\varphi\rangle \\ &= 2\langle \Pi_\gamma\partial_{\omega_\gamma}\varphi, \varphi\rangle + 2\langle \Pi_\gamma\varphi, \partial_{\omega_\gamma}\varphi\rangle = 4\langle \Pi_\gamma\varphi, \partial_{\omega_\gamma}\varphi\rangle.\end{aligned} \tag{4.2.5}$$

Next, we will compute $\partial_{\omega_\gamma}\varphi$.
Differentiating both sides of (4.2.1) with respect to ω_γ to get

$$\begin{aligned}\big(\partial_{\omega_\gamma}H_\omega(\Lambda)\big)\varphi + H_\omega(\Lambda)\partial_{\omega_\gamma}\varphi &= \partial_{\omega_\gamma}E(\omega)\varphi + E(\omega)\partial_{\omega_\gamma}\varphi \\ &= 2\langle \Pi_\gamma\varphi, \varphi\rangle\varphi + E(\omega)\partial_{\omega_\gamma}\varphi.\end{aligned}$$

Therefore,

$$[H_\omega(\Lambda) - E(\omega)]\partial_{\omega_\gamma}\varphi = 2\langle \Pi_\gamma\varphi, \varphi\rangle\varphi - \big(\partial_{\omega_\gamma}H_\omega(\Lambda)\big)\varphi = 2\Big(\langle \Pi_\gamma\varphi, \varphi\rangle\varphi - \Pi_\gamma\varphi\Big).$$

Observe that $\psi_\gamma := \langle \Pi_\gamma\varphi, \varphi\rangle\varphi - \Pi_\gamma\varphi \in \langle\varphi\rangle^\perp$, and $[H_\omega(\Lambda) - E(\omega)]$ is invertible in the subspace $\langle\varphi\rangle^\perp$ of $l^2(\Lambda)$, we get

$$\partial_{\omega_\gamma}\varphi = 2\big(H_\omega(\Lambda) - E(\omega)\big)^{-1}\big(\langle \Pi_\gamma\varphi, \varphi\rangle\varphi - \Pi_\gamma\varphi\big). \tag{4.2.6}$$

From (4.2.5) and (4.2.6), we obtain

$$\partial^2_{\omega_\gamma}E(\omega) = 4\big\langle \Pi_\gamma\varphi, \big(H_\omega(\Lambda) - E(\omega)\big)^{-1}\big(\langle \Pi_\gamma\varphi, \varphi\rangle\varphi - \Pi_\gamma\varphi\big)\big\rangle.$$

Thanks to (4.2.6), we have $2\big(H_\omega(\Lambda)-E(\omega)\big)^{-1}\big(\langle\Pi_\gamma\varphi,\varphi\rangle\varphi-\Pi_\gamma\varphi\big)$ is orthogonal to φ. We therefore infer that

$$\partial^2_{\omega_\gamma}E(\omega)=-4\Big\langle\big(H_\omega(\Lambda)-E(\omega)\big)^{-1}\psi_\gamma,\psi_\gamma\Big\rangle.$$

Repeating this argument, it is not hard to prove that

$$\partial^2_{\omega_\gamma\omega_\beta}E(\omega)=-4\Big\langle\big(H_\omega(\Lambda)-E(\omega)\big)^{-1}\psi_\gamma,\psi_\beta\Big\rangle$$

for all γ,β. So, we have Lemma 4.2.1 proved. □

Assume that $E(\omega)$ is an eigenvalue of $H_\omega(\Lambda)$ with $\Lambda=[-L,L]$. Recall that $\omega_j\in[\alpha_0,\beta_0]$ for all $j\in\mathbb{Z}$. From Lemma 4.2.1, we have $E(\omega)\in[0,4\beta_0]$. Denote by $u:=u(\omega)$ the normalized eigenvector associated to $E(\omega)$. We would like to prove a "lower bound" for u in the sense that there exists a large subset J in Λ such that the components $(u(k))_{k\in J}$ of u can not be too small.

Lemma 4.2.2. *Pick $\beta\in(1/2,1)$. Then, there exists a point k_0 in Λ and a positive constant κ depending only on α_0,β_0 such that*

$$u^2(k)+u^2(k+1)\geqslant e^{-L^\beta/2}$$

for all $|k-k_0|\leqslant\kappa L^\beta$ when L is large enough.

Proof of Lemma 4.2.2. We rewrite the eigenequation corresponding to the eigenvector u and eigenvalue $E(\omega)$ at the point n by means of the transfer matrix

$$\begin{pmatrix}u(n+1)\\u(n)\end{pmatrix}=\begin{pmatrix}\dfrac{\omega_n+\omega_{n-1}-E(\omega)}{\omega_n} & \dfrac{-\omega_{n-1}}{\omega_n}\\ 1 & 0\end{pmatrix}\begin{pmatrix}u(n)\\u(n-1)\end{pmatrix}.$$

Now, let $T(n,E(\omega))$ and $v(n)$ denote the transfer matrix

$$\begin{pmatrix}\dfrac{\omega_n+\omega_{n-1}-E(\omega)}{\omega_n} & \dfrac{-\omega_{n-1}}{\omega_n}\\ 1 & 0\end{pmatrix}$$

and the column vector $(u(n+1),u(n))^t$ respectively.
Then, for all n greater than m we have

$$v(n)=T(n,E(\omega))\cdots T(n-m+1,E(\omega))v(m).$$

It is easy to check that the transfer matrices are invertible. Moreover, since $E(\omega) \in [0, 4\beta_0]$ and $0 < \alpha_0 \leq \omega_j \leq \beta_0$, they and their inverse matrices are uniformly bounded by a constant $C > 1$ depending only on α_0 and β_0.
Thus,

$$\|v(n)\| \leqslant C^{|n-m|}\|v(m)\| = e^{(\log C)|n-m|}\|v(m)\| = e^{\eta|n-m|}\|v(m)\|$$

for all n, m in Λ with $\eta = \log C > 0$.
Assume that $\|v(k_0)\|$ is the maximum of $\|v(n)\|$. Hence,

$$\|v(k_0)\| \geqslant \frac{1}{\sqrt{2L}}$$

from the fact that $\sum_{j\in\Lambda} \|v(j)\|^2 = 2$. Thus, for any $\kappa > 0$, the following holds true

$$\|v(k)\| \geqslant \frac{1}{\sqrt{2L}} e^{-\eta|k-k_0|} \geq e^{-2\kappa\eta L^\beta}$$

for $|k - k_0| \leqslant \kappa L^\beta$ and L sufficiently large.
In other words, we have

$$u^2(k) + u^2(k+1) \geqslant e^{-4\kappa\eta L^\beta}$$

for all $|k - k_0| \leqslant \kappa L^\beta$. So, by choosing $\kappa = \dfrac{1}{8\eta}$, we have Lemma 4.2.2 proved. □

The following lemma is the main ingredient of the proof of the decorrelation estimate as well as the heart of the present chapter:

Lemma 4.2.3. *Let $E \neq E'$ be two positive energies in the localized regime and $\beta \in (1/2, 1)$. Assume that $\Lambda = \Lambda_L =: [-L, L]$ is a large interval in $\mathbb{Z}$. Pick $c_1, c_2 > 0$ and denote by $\mathbb{P}^*$ the probability of the following event (called (*)):*
there exist two simple eigenvalues of $H_\omega(\Lambda)$, say $E(\omega), E'(\omega)$ such that $|E(\omega)-E|+|E'(\omega)-E'| \leqslant e^{-L^\beta}$ and

$$\|\nabla_\omega(c_1 E(\omega) - c_2 E'(\omega))\|_1 \leqslant c_1 e^{-L^\beta}.$$

Then, there exists $c > 0$ such that

$$\mathbb{P}^* \leqslant e^{-cL^{2\beta}}.$$

Remark 4.2.4. *There is a slightly difference between the above lemma and Lemma 2.4 in [Klo11] where $c_1 = c_2 = 1$. In fact, in the proof of Theorem 4.1.2, we will use the above lemma with c_1, c_2 are respectively $\frac{1}{E}, \frac{1}{E'}$ which are two distinct, positive numbers. This difference results from the lack of the normalization of $\|\nabla E(\omega)\|_1$ for our model. Moreover, we will see in Remark 4.2.12 at the end of this section that, for the model (4.1.1), if $c_1 = c_2$, $\mathbb{P}^*$ is equal to 0 for all L large.*

We will skip for a moment the proof of Lemma 4.2.3 and recall how to use the above lemma to complete the proof of Theorem 4.1.2. This part can be found in [Klo11]. We repeat it here with tiny but necessary changes adapted for the model (4.1.1).

Proof of Theorem 4.1.2. Recall that E, E' are two positive, fixed energies and $J_L = E + L^{-1}[-1,1]$ and $J'_L = E' + L^{-1}[-1,1]$. One chooses L large enough such that $\min\{E - L^{-1}, E' - L^{-1}\} \geq \min\{E, E'\}/2 > 0$.
Let $cL^\alpha \leqslant l \leqslant L^\alpha/c$ with $c > 0$. From the Minami estimate, one has

$$\begin{aligned}&\mathbb{P}\big(\#\{\sigma(H_\omega(\Lambda_l)) \cap J_L\} \geqslant 2 \text{ or } \#\{\sigma(H_\omega(\Lambda_l)) \cap J'_L\} \geqslant 2\big)\\ &\leqslant \frac{4\beta_0\|\rho\|_\infty \|s\rho(s)\|_\infty}{(\min\{E, E'\})^2}(|\Lambda_l||J_L|)^2 \leq C(l/L)^2\end{aligned}$$

where C is a constant depending only on E, E', β_0 and ρ.
Hence, it is sufficient to prove that $\mathbb{P}_0 \leqslant C(l/L)^2 e^{(\log L)^\beta}$ where

$$\mathbb{P}_0 := \mathbb{P}\big(\#\{\sigma(H_\omega(\Lambda_l)) \cap J_L\} = 1;\ \#\{\sigma(H_\omega(\Lambda_l)) \cap J'_L\} = 1\big). \tag{4.2.7}$$

The crucial idea of proving decorrelation estimates in [Klo11] is to reduce the proof of (4.2.7) to the proof of a similar estimate where Λ_l is replaced by a much smaller cube, a cube of side length of order $\log L$. Precisely, one has (c.f. Lemma 2.1 and Lemma 2.2 in [Klo11]):

$$\mathbb{P}_0 \leqslant C(l/L)^2 + C(l/\widetilde{l})\mathbb{P}_1$$

where $\widetilde{l} = C\log L$ and

$$\mathbb{P}_1 := \mathbb{P}\big(\#[\sigma(H_\omega(\Lambda_{\widetilde{l}})) \cap \widetilde{J}_L] \geqslant 1 \text{ and } \#[\sigma(H_\omega(\Lambda_{\widetilde{l}})) \cap \widetilde{J}'_L] \geqslant 1\big)$$

where $\widetilde{J}_L = E + L^{-1}(-2,2)$ and $\widetilde{J}'_L = E' + L^{-1}(-2,2)$.
To complete the proof of Theorem 4.1.2, one need show that

$$\mathbb{P}_1 \leqslant C(\widetilde{l}/L)^2 e^{\widetilde{l}^\beta}. \tag{4.2.8}$$

Thanks to the Minami estimate and the following inequality (see Lemma 2.3 in [Klo11] for a proof)

$$\|Hess_\omega(E(\omega))\|_{l^\infty \to l^1} \leqslant \frac{C}{dist\big(E(\omega), \sigma(H_\omega(\Lambda)) \setminus E(\omega)\big)}, \tag{4.2.9}$$

one infers that

$$\mathbb{P}\Big(\begin{Bmatrix}\sigma(H_\omega(\Lambda_{\widetilde{l}})) \cap (\widetilde{J}_L) = \{E(\omega)\}\\ \|Hess_\omega(E(\omega))\|_{l^\infty \to l^1} \geqslant \epsilon^{-1}\end{Bmatrix}\Big) \leqslant C\epsilon\widetilde{l}^2 L^{-1}.$$

Hence, for $\epsilon \in (4L^{-1}, 1)$, one has

$$\mathbb{P}_1 \leqslant C\epsilon\widetilde{l}^2 L^{-1} + \mathbb{P}_\epsilon \tag{4.2.10}$$

where $\mathbb{P}_\epsilon = \mathbb{P}(\Omega_0(\epsilon))$ with

$$\Omega_0(\epsilon) := \left\{ \begin{array}{c} \sigma(H_\omega(\Lambda_{\widetilde{l}})) \cap \widetilde{J}_L = \{E(\omega)\} \\ \{E(\omega)\} = \sigma(H_\omega(\Lambda_{\widetilde{l}})) \cap (E - C\epsilon, E + C\epsilon) \\ \sigma(H_\omega(\Lambda_{\widetilde{l}})) \cap \widetilde{J}'_L = \{E'(\omega)\} \\ \{E'(\omega)\} = \sigma(H_\omega(\Lambda_{\widetilde{l}})) \cap (E' - C\epsilon, E' + C\epsilon) \end{array} \right\}.$$

Next, one puts $\lambda := e^{-\widetilde{l}^\beta}$ and defines, for $\gamma, \gamma' \in \Lambda_{\widetilde{l}}$,

$$\Omega_{0,\beta}^{\gamma,\gamma'}(\epsilon) := \Omega_0(\epsilon) \cap \{\omega \mid |J_{\gamma,\gamma'}(E(\omega), E'(\omega))| \geqslant \lambda\}$$

where $J_{\gamma,\gamma'}(E(\omega), E'(\omega))$ is the Jacobian of the mapping

$$(\omega_\gamma, \omega_{\gamma'}) \mapsto (E(\omega), E'(\omega)).$$

On the one hand, $\mathbb{P}_\epsilon$ can be dominated as follows:

$$\mathbb{P}_\epsilon \leqslant \sum_{\gamma \neq \gamma'} \mathbb{P}(\Omega_{0,\beta}^{\gamma,\gamma'}(\epsilon)) + \mathbb{P}_r$$

where $\mathbb{P}_r$ is the probability of the following event

$$\mathcal{D} := \{\omega \in \Omega_0(\epsilon) \mid |J_{\gamma,\gamma'}(E(\omega), E'(\omega))| \leqslant \lambda \text{ for all } \gamma, \gamma' \in \Lambda_{\widetilde{l}}\}.$$

On the other hand, from Lemma 2.6 in [Klo11], it is known that

$$\mathbb{P}(\Omega_{0,\beta}^{\gamma,\gamma'}(\epsilon)) \leqslant CL^{-2}\lambda^{-4} \text{ for all } \gamma, \gamma' \in \Lambda_{\widetilde{l}}.$$

Hence,

$$\mathbb{P}_\epsilon \leqslant C\widetilde{l}^2 L^{-2}\lambda^{-4} + \mathbb{P}_r. \tag{4.2.11}$$

Choose $\epsilon := L^{-1}\lambda^{-3}$, (4.2.10) and (4.2.11) yield that

$$\mathbb{P}_1 \leqslant C(\widetilde{l}/L)^2 e^{\widetilde{l}^\beta} + \mathbb{P}_r. \tag{4.2.12}$$

Finally, we will use Lemma 4.2.3 to estimate $\mathbb{P}_r$.
For each $\omega \in \mathcal{D}$, we rewrite the Jacobian $J_{\gamma,\gamma'}(E(\omega), E'(\omega))$ as follows:

$$J_{\gamma,\gamma'}(E(\omega), E'(\omega)) = \begin{vmatrix} \partial_{\omega_\gamma} E(\omega) & \partial_{\omega_{\gamma'}} E(\omega) \\ \partial_{\omega_\gamma} E'(\omega) & \partial_{\omega_{\gamma'}} E'(\omega) \end{vmatrix}$$

$$= \frac{E(\omega)E'(\omega)}{\omega_\gamma \omega_{\gamma'}} \begin{vmatrix} \frac{1}{E(\omega)}\omega_\gamma \partial_{\omega_\gamma} E(\omega) & \frac{1}{E(\omega)}\omega_{\gamma'} \partial_{\omega_{\gamma'}} E(\omega) \\ \frac{1}{E'(\omega)}\omega_\gamma \partial_{\omega_\gamma} E'(\omega) & \frac{1}{E'(\omega)}\omega_{\gamma'} \partial_{\omega_{\gamma'}} E'(\omega) \end{vmatrix} \tag{4.2.13}$$

if $E(\omega)$ and $E'(\omega)$ are non-zero.

Note that, from (4.2.3), one has

$$\sum_{\gamma\in\Lambda_{\tilde{l}}}\frac{1}{E(\omega)}\omega_\gamma\partial_{\omega_\gamma}E(\omega)=\sum_{\gamma\in\Lambda_{\tilde{l}}}\frac{1}{E'(\omega)}\omega_\gamma\partial_{\omega_\gamma}E'(\omega)=1.$$

Hence, one can apply Lemma 2.5 in [Klo11] to (4.2.13) and deduce that

$$\|\nabla_\omega(\frac{1}{E}E(\omega)-\frac{1}{E'}E'(\omega))\|_1\leqslant e^{-\tilde{l}^{\beta'}}$$

for any $1/2<\beta'<\beta$.

Thus, Lemma 4.2.3 yields that, for L sufficiently large,

$$\mathbb{P}_r\leqslant\tilde{l}^2e^{-c\tilde{l}^{2\beta'}}=O(L^{-\infty}). \tag{4.2.14}$$

From (4.2.12) and (4.2.14), (4.2.8) follows and we have Theorem 4.1.2 proved. □

Before coming to the proof of Lemma 4.2.3, we state and prove here a short lemma which will be used repeatedly in the rest of this section.

Lemma 4.2.5. *Pick $A\in Mat_n(\mathbb{R})$ and $b\in\mathbb{R}^n$ such that $\|b\|\leq c_0e^{-L^\beta/2}$ where c_0, L are fixed, positive constants. Assume that the following system of linear equations*

$$Ax=b$$

has a solution u satisfying $\|u\|\geq e^{-L^\beta/4}$.
Then,

$$|\det A|\leq c_0\max\{1,||adj(A)||\}e^{-L^\beta/4}$$

where $adj(A)$ is the adjugate (the transpose of the cofactor matrix) of A.

Proof of Lemma 4.2.5. Assume by contradiction that

$$|\det A|>c_0\max\{1,||\text{adj}(A)||\}e^{-L^\beta/4}>0.$$

Consequently, A is invertible and $u=A^{-1}b$ is the unique solution of the system $Ax=b$. We therefore infer that

$$\|u\|\leq\|A^{-1}\|\|b\|=\frac{1}{|\det A|}\|\text{adj}(A)\|\|b\|\leq\frac{\max\{1,||\text{adj}(A)||\}}{|\det A|}\|b\|.$$

Hence,

$$|\det A|\leq c_0\max\{1,||\text{adj}(A)||\}e^{-L^\beta/2}e^{L^\beta/4}=c_0\max\{1,||\text{adj}(A)||\}e^{-L^\beta/4}$$

which is a contradiction. □

To complete the present section, we state here the proof of Lemma 4.2.3.

Proof of Lemma 4.2.3. Let $u := u(\omega)$ and $v := v(\omega)$ be normalized eigenvectors associated to $E(\omega)$ and $E'(\omega)$. By Lemma 4.2.1, we have

$$\nabla_\omega E(\omega) = \left(2\|\Pi_\gamma u\|^2\right)_{\gamma\in\Lambda} \text{ and } \nabla_\omega E'(\omega) = \left(2\|\Pi_\gamma v\|^2\right)_{\gamma\in\Lambda}$$

We introduce the linear operator T from $l^2(\Lambda)$ to $l^2(\Lambda)$ defined as follows

$$Tu(n) = u(n) - u(n+1)$$

where $u = (u(n))_n \in l^2(\Lambda)$. Recall that $\Lambda = \Lambda_L = \mathbb{Z}/L\mathbb{Z}$, i.e. we use periodic boundary conditions here.

Assume that $\{\omega_j\}_{j\in\Lambda}$ belongs to the event (*). We thus have

$$\begin{aligned} c_1 e^{-L^\beta} &\geqslant \|\nabla_\omega(c_1 E(\omega) - c_2 E'(\omega))\|_1 = \sum_n |(\sqrt{c_1}Tu(n))^2 - (\sqrt{c_2}Tv(n))^2| \\ &= \sum_n |\sqrt{c_1}Tu(n) - \sqrt{c_2}Tv(n)||\sqrt{c_1}Tu(n) + \sqrt{c_2}Tv(n)|. \end{aligned}$$

Hence, $e^{-L^\beta} \geqslant \sum_n |Tu(n) - cTv(n)||Tu(n) + cTv(n)|$ with $c = \sqrt{c_2}/\sqrt{c_1} > 0$.

Then, there exists a partition of $\Lambda = \{-L, \dots, L\}$, say $\mathcal{P} \subset \Lambda$ and $\mathcal{Q} \subset \Lambda$ such that $\mathcal{P} \cup \mathcal{Q} = \Lambda$, $\mathcal{P} \cap \mathcal{Q} = \emptyset$ and

- for $n \in \mathcal{P}$, $|Tu(n) - cTv(n)| \leqslant e^{-L^\beta/2}$,
- for $n \in \mathcal{Q}$, $|Tu(n) + cTv(n)| \leqslant e^{-L^\beta/2}$.

From now on, we put $v(n) := cv(n)$. This abuse of notation changes nothing thanks to the linearity of the operator T.

Hence, we obtain that

$$\begin{cases} |Tu(n) = Tv(n) + \mathrm{O}(e^{-L^\beta/2}) & \text{if } n \in \mathcal{P}, \\ |Tu(n) = -Tv(n) + \mathrm{O}(e^{-L^\beta/2}) & \text{if } n \in \mathcal{Q}. \end{cases} \tag{4.2.15}$$

From Lemma 4.2.2, there exists $c_3 > 0$ depending only on α_0, β_0 and an interval J of the length $c_3 L^\beta$ such that

$$|u(k)|^2 + |u(k+1)|^2 \geqslant 2e^{-L^\beta/2} \tag{4.2.16}$$

for all $k \in J$.

Now, we decompose

$$\mathcal{P} \cap J = \cup \mathcal{P}_j \text{ and } \mathcal{Q} \cap J = \cup \mathcal{Q}_j \tag{4.2.17}$$

where $\mathcal{P}_j$ and $\mathcal{Q}_j$ are intervals in $\mathbb{Z}$.
We will divide the rest of the proof into some lemmata. First of all, in the Lemma 4.2.6, we show a restriction on the length of each interval $\mathcal{P}_j$ and $\mathcal{Q}_j$ in $\mathbb{Z}$. We will make use of this lemma later to prove a "reduction" lemma (Lemma 4.2.10). Next, In Lemma 4.2.8, with any four consecutive points in J, we explain how to form an inhomogeneous 10×10 system of linear equations from (4.2.15) and eigenequations for u and v. Finally, we show some restrictions on $\{\omega_j\}_{j\in\Lambda}$ in Lemma 4.2.11. Thanks to this lemma and Lemma 4.2.10, Lemma 4.2.3 follows.

Lemma 4.2.6. *Assume that $\{\omega_j\}_{j\in\Lambda}$ belongs to the event (*) defined in Lemma 4.2.3: $H_\omega(\Lambda)$ has two simple eigenvalues $E(\omega), E'(\omega)$ such that $|E(\omega)-E|+|E'(\omega)-E'| \leqslant e^{-L^\beta}$ and*

$$\|\nabla_\omega(c_1E(\omega)-c_2E'(\omega))\|_1 \leqslant c_1e^{-L^\beta}.$$

Denote by u, v normalized eigenvectors associated to $E(\omega), E'(\omega)$ respectively and consider the decomposition $\{\mathcal{P}_i, \mathcal{Q}_j\}$ in (4.2.17). Then, any $\mathcal{P}_j$ or $\mathcal{Q}_j$ can not contain more than four points.

Proof of Lemma 4.2.6. Thanks to the equivalent role of $\mathcal{P}$ and $\mathcal{Q}$, it is sufficient to prove Lemma 4.2.6 for $\{\mathcal{P}_j\}_j$.
Assume by contradiction that there exists an interval $\mathcal{P}_j$ contain at least five consecutive points, say $\mathcal{P}_j = \{n-2, n-1, n, n+1, n+2, \dots, m\}$ with $m \geqslant n+2$.
First of all, thanks to (4.2.15), we have

$$\begin{aligned} Tu(n-2) &= Tv(n-2) + O(e^{-L^\beta/2}). \\ Tu(n-1) &= Tv(n-1) + O(e^{-L^\beta/2}). \\ Tu(n) &= Tv(n) + O(e^{-L^\beta/2}). \\ Tu(n+1) &= Tv(n+1) + O(e^{-L^\beta/2}). \\ Tu(n+2) &= Tv(n+2) + O(e^{-L^\beta/2}). \end{aligned} \tag{4.2.18}$$

Next, consider the triple of consecutive points $\{n-2, n-1, n\} \in \mathcal{P}_j$. Using the eigenequations for u and v at the point $(n-1)$ and take the hypothesis $|E(\omega)-E|+|E'(\omega)-E'| \leq e^{-L^\beta}$ into account, we deduce

$$Eu(n-1) = \omega_{n-1}Tu(n-1) - \omega_{n-2}Tu(n-2) + O(e^{-L^\beta/2}), \tag{4.2.19}$$

$$E'v(n-1) = \omega_{n-1}Tv(n-1) - \omega_{n-2}Tv(n-2) + O(e^{-L^\beta/2}). \tag{4.2.20}$$

Hence, (4.2.19), (4.2.20) and the first two equations in (4.2.18) yield

$$Eu(n-1) = E'v(n-1) + O(e^{-L^\beta/2}). \tag{4.2.21}$$

Similarly, we have

$$Eu(n) = E'v(n) + O(e^{-L^\beta/2}). \tag{4.2.22}$$

Combining (4.2.21), (4.2.22) and the second equation in (4.2.18), we obtain

$$\Big(1 - \frac{E}{E'}\Big)u(n) = \Big(1 - \frac{E}{E'}\Big)u(n-1) + O(e^{-L^\beta/2})$$

which implies that

$$Tu(n-1) \leq Ce^{-L^\beta/2} \tag{4.2.23}$$

where C is a positive constant depending only on E, E', α_0 and β_0.
Repeating again the above argument for other triples of consecutive points in $\mathcal{P}_j$, we obtain

$$Tu(n) \leq Ce^{-L^\beta/2} \tag{4.2.24}$$

and

$$Tu(n+1) \leq Ce^{-L^\beta/2}. \tag{4.2.25}$$

On the other hand, we have the following eigenequations for u at the point n and $n+1$

$$Eu(n) = \omega_n Tu(n) - \omega_{n-1}Tu(n-1) + O(e^{-L^\beta/2}),$$
$$Eu(n+1) = \omega_{n+1} Tu(n+1) - \omega_n Tu(n) + O(e^{-L^\beta/2}).$$

Hence, combining the above equations and (4.2.23)-(4.2.25), we infer that there exists a positive constant C being independent of L such that

$$|u(n)|^2 + |u(n+1)|^2 \leqslant Ce^{-L^\beta}$$

which contradicts (4.2.16) if we choose L large enough.
Hence, an interval $\mathcal{P}_j$ or $\mathcal{Q}_j$ can not contain more than four points in $\mathbb{Z}$ and we have Lemma 4.2.6 proved. □

From the proof of Lemma 4.2.6, we reach to the following conclusion:

Remark 4.2.7. *If two consecutive points ordered increasingly belong to some interval $\mathcal{P}_j$ ($n-2, n-1$ for instance), the value of u at the latter point ($n-1$ in this case) is proportional to the value of v at that point (as in (4.2.21)) up to an exponentially small error. Moreover,* *if we have three consecutive points ordered increasingly in some interval $\mathcal{P}_j$ $(n-2, n-1, n)$, the middle point $(n-1)$ always satisfies an inequality of the form (4.2.23). Finally, if three points $n-2, n-1, n$ belong to some $\mathcal{Q}_i$, we will have almost the same conclusion except that E' need replacing by $-E'$ in (4.2.21) and (4.2.22).*

Lemma 4.2.8. *Let J be the subinterval of Λ where (4.2.16) holds and $n-2, n-1, n, n+1$ be four consecutive points in J. Assume the same hypotheses as in Lemma 4.2.6 and put $U := (u(n-2), \dots, u(n+2), v(n-2), \dots, v(n+2))^t$. Then, from (4.2.15) and the eigenequations for u and v, we can form a 10×10 system of linear equations which admits U as one of its solutions.*

Proof of Lemma 4.2.8. From (4.2.15), for each of these four points, we have an equation of the form

$$Tu(k) = \pm Tv(k) + O(e^{-L^\beta/2}),\ k = \overline{n-2, n+1} \tag{4.2.26}$$

where the choice of $(+)$ or $(-)$ sign depends on whether k belongs to $\mathcal{P}$ or $\mathcal{Q}$. So, we have 4 (inhomogeneous) linear equations in hand.
On the other hand, we have 6 eigenequations of eigenvectors u and v at the points $n-1, n$ and $n+1$. Hence, apparently, we have 10 linear equations corresponding to 10 variables $\{u(n-2), \dots, u(n+2), v(n-2), \dots, v(n+2)\}$. However, there is a couple of things here which should be made clearer.
First of all, to form our systems of linear equations, we use the following three eigenequations w.r.t. u

$$Eu(k) = \omega_k Tu(k) - \omega_{k-1} Tu(k-1) + O(e^{-L^\beta/2}) \tag{4.2.27}$$

where $k = \overline{n-1, n+1}$.
Next, we consider the eigenequations of v at $k = \overline{n-1, n+1}$

$$E'v(k) = \omega_k Tv(k) - \omega_{k-1} Tv(k-1) + O(e^{-L^\beta/2}). \tag{4.2.28}$$

Instead of using directly (4.2.28) for our 10×10 systems of linear equations, we substitute (4.2.26) into the right hand side of (4.2.28) to get

$$E'v(k) = \pm\omega_k Tu(k) \mp \omega_{k-1} Tu(k-1) + O(e^{-L^\beta/2}) \tag{4.2.29}$$

and (4.2.29) will be used for our 10×10 systems of linear equations.
In Lemma 4.2.11, we will write down these 10×10 systems of linear equations as follows: The first four equations come from (4.2.26). Then, we write down the equations in (4.2.27) and (4.2.29). The fifth equation is (4.2.27) and the sixth is (4.2.29) with $k = n$. The

seventh is (4.2.27) and the eighth is (4.2.29) with $k = n + 1$. Lastly, (4.2.27) and (4.2.29) with $k = n - 1$ are the ninth and the tenth equation in turn.
Finally, we make an important remark in case there exists an interval $\mathcal{P}_j$ or $\mathcal{Q}_j$ contains at least two consecutive points of these four points, says $j-1, j$. According to Remark 4.2.7, we have

$$v(j) = \pm \frac{E}{E'} u(j) + O(e^{-L^{\beta}/2}) \tag{4.2.30}$$

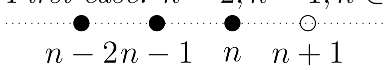
where (4.2.30) takes $(+)$ sign iff $j-1$ and $j \in \mathcal{P}$.
Whenever (4.2.30) holds true, we will use it to replace (4.2.29) w.r.t. $k = j$ in our systems of linear equations. This replacement simplifies these 10×10 systems of linear equations and makes them easier to analyze. □

Definition 4.2.9. *A point $n \in J$ is an interior point of J if the interval $[n-2, n+2]$ belongs to J.*

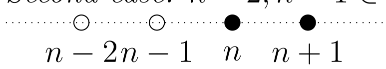
Let $n-2, n-1, n, n+1$ be interior points in J. We consider all possible 10×10 systems of linear equations which we can get from these points as in Lemma 4.2.8. We have four points, each of them can belong to $\mathcal{P}$ or $\mathcal{Q}$. Hence, the number of choices for four points' belonging to $\mathcal{P}$ or $\mathcal{Q}$ equals $2^4 = 16$ which is also the total number of 10×10 systems of linear equations obtained in Lemma 4.2.8. Furthermore, we have the following useful observation:

Lemma 4.2.10. *Assume the same hypotheses as in Lemma 4.2.6. Let $n-2, n-1, n, n+1$ be interior points in J and $\{\mathcal{P}_i, \mathcal{Q}_j\}$ be the decomposition in (4.2.17). Then, we will only need to analyze 10×10 systems of linear equations corresponding to the following four cases:*

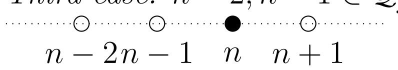
First case: $n-2, n-1, n \in \mathcal{P}_j$ and $n+1 \in \mathcal{Q}_j$,



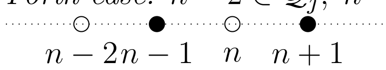
Second case: $n-2, n-1 \in \mathcal{Q}_j$ and $n, n+1 \in \mathcal{P}_j$,



Third case: $n-2, n-1 \in \mathcal{Q}_j$, $n \in \mathcal{P}_j$ and $n+1 \in \mathcal{Q}_{j+1}$,

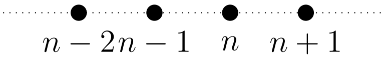


Forth case: $n-2 \in \mathcal{Q}_j$, $n-1 \in \mathcal{P}_j$, $n \in \mathcal{Q}_{j+1}$ and $n+1 \in \mathcal{P}_{j+1}$.



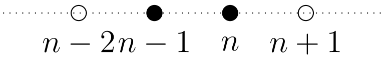
Proof of Lemma 4.2.10. As mentioned above, we have a total of 16 systems of linear equations to analyze. Thanks to the equivalent role of $\mathcal{P}$ and $\mathcal{Q}$, we only have to consider a half of them. Apart from four cases listed above, the other cases are:

Fifth case: Assume that all of these four points belong to some $\mathcal{P}_j$.



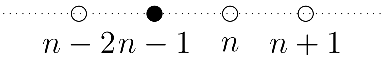
Hence, From Lemma 4.2.6, $n+2 \in \mathcal{Q}_j$. We consider four points $n-1, n, n+1, n+2$ and come back to *First case*.

Sixth case: Suppose that $n-2 \in \mathcal{Q}_j$, $n-1, n \in \mathcal{P}_j$, and $n+1 \in \mathcal{Q}_{j+1}$.



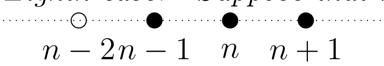
We consider the point $n+2$. If $n+2$ belongs to $\mathcal{Q}_{j+1}$, we consider four points $n-1, n, n+1, n+2$ and come back to *Second case* because of the equivalent role of $\mathcal{P}$ and $\mathcal{Q}$. Otherwise, $n+2$ belongs to $\mathcal{P}_{j+1}$. In this case, we consider four points $n-1, n, n+1, n+2$ and come back to *Third case*.

Seventh case: Assume that $n-2 \in \mathcal{Q}_j$, $n-1 \in \mathcal{P}_j$ and $n, n+1 \in \mathcal{Q}_{j+1}$.



In this case, we consider four points $n-3, n-2, n-1, n$. If $n-3$ also belongs to $\mathcal{Q}_j$, we come back to *Third case*. Otherwise, we come back to *Forth case* on account of the equivalent role of $\mathcal{P}$ and $\mathcal{Q}$.

Eighth case: Suppose that $n-2 \in Q_j$ and $n-1, n, n+1 \in \mathcal{P}_j$ for some j.



If $n+2 \in \mathcal{Q}_{j+1}$, we consider four points $n-1, n, n+1, n+2$ and come back to *First case*. Otherwise, $n+2$ still belongs to $\mathcal{P}_j$, hence $n+3 \in \mathcal{Q}_{j+1}$ according to Lemma 4.2.6. On the other hand, $n+3$ still belongs to J since $n+1$ is the interior point of J. Hence, we consider four points $n, n+1, n+2, n+3$ and come back to *First case*.

To conclude, we only need to analyze 4 special cases. The other cases can be reduced to those ones. □

Now, we come to the final stage in the proof of Lemma 4.2.3 where we deduce the restrictions on r.v.'s ω_j.

Lemma 4.2.11. *Assume that hypotheses of Lemma 4.2.6 hold. Let J be the interval defined in (4.2.15) and $n-2, n-1, n, n+1$ be four interior points of J. Assume that these four points correspond to one of the four cases listed in Lemma 4.2.10. Then, one of the following restrictions on r.v.'s holds true* *(i)* $\left|\omega_n - \dfrac{E+E'}{4}\right| \leqslant Ce^{-L^\beta/8}$,

(ii) $\left|\omega_{n-1} - \dfrac{E'+E}{4}\right| \leqslant Ce^{-L^\beta/8}$,

(iii) $\left|\omega_{n-1}\omega_n - \dfrac{(E-E')^2}{4}\right| \leqslant Ce^{-L^\beta/4}$.

Proof of Lemma 4.2.11. For each of four cases in Lemma 4.2.10, we consider the corresponding system of linear equations formed in Lemma 4.2.8 and compute its determinant. This yields some restrictions on r.v.'s.
Recall that $U := (u(n-2),\dots,u(n+2),v(n-2),\dots,v(n+2))^t$.
First case: Assume that three points $n-2, n-1, n \in \mathcal{P}_j$ and the other one $n+1 \in \mathcal{Q}_j$.
Since $n-2, n-1, n \in \mathcal{P}_j$, two equations in (4.2.29) associated to $n-1, n$ will be replaced by two equations of the type (4.2.30) with $(+)$ sign in our system. Hence, according to Lemma 4.2.8, U satisfies the following system of linear equations:

$$A_0 U = b_0 \tag{4.2.31}$$

where $b_0 = (b_0^j)_{1\leqslant j\leqslant 10}$ with $\|b_0\| \leqslant e^{-L^\beta/2}$ and A_0 is the 10×10 matrix of the block form $(A_0^1|A_0^2)$ with

$$A_0^1 := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -\omega_{n-1} & \omega_{n-1}+\omega_n-E & -\omega_n & 0 \\ 0 & 0 & \dfrac{E}{E'} & 0 & 0 \\ 0 & 0 & -\omega_n & \omega_n+\omega_{n+1}-E & -\omega_{n+1} \\ 0 & 0 & \omega_n & \omega_{n+1}-\omega_n & -\omega_{n+1} \\ -\omega_{n-2} & \omega_{n-2}+\omega_{n-1}-E & -\omega_{n-1} & 0 & 0 \\ 0 & \dfrac{E}{E'} & 0 & 0 & 0 \end{pmatrix}$$

and

$$A_0^2 := \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E' & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\text{adj}(A_0)$ be the adjugate of A_0. It is easy to see that

$$\max\{1, \|\text{adj}(A_0)\|\} \leqslant M_0$$

where M_0 is a positive constant depending only on E, E', α_0, β_0.
Hence, thanks to Lemma 4.2.5, we have

$$|\det A_0| \leq M_0 e^{-L^\beta/4}.$$

By an explicit computation given in Appendix A, we have

$$|\det A_0| = \frac{4E}{E'}|E - E'|\omega_{n-2}\omega_{n+1}\left|\omega_n - \frac{E' + E}{4}\right| \leqslant M_0 e^{-L^\beta/4}.$$

Therefore, from the fact that $E, E' > 0$ and $\omega_j \geq \alpha_0 > 0$, the following condition on ω holds true with L sufficiently large:

$$\left|\omega_n - \frac{E' + E}{4}\right| \leqslant C e^{-L^\beta/4}. \qquad (I)$$

Second case: $n-2, n-1 \in \mathcal{Q}_j$ *and* $n, n+1 \in \mathcal{P}_j$.
In the present case, since $n-2, n-1 \in \mathcal{Q}_j$, we use (4.2.30) with (-) sign w.r.t. $n-1$ to replace the equation in (4.2.29) w.r.t. $n-1$ for our system of linear equations. Besides, since $n, n+1 \in \mathcal{P}_j$, the equation in (4.2.29) w.r.t. $n+1$ will be replaced by (4.2.30) with $(+)$ sign at $n+1$.
Hence, according to Lemma 4.2.8, we have the following 10×10 system of linear equations:

$$A_1 U = b_1 \qquad (4.2.32)$$

where $b_1 = (b_1^j)_{1\leqslant j\leqslant 10}$, $\|b_1\| \leqslant e^{-L^\beta/2}$ and $A_1 = (A_1^1|A_1^2)$ with

$$A_1^1 := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -\omega_{n-1} & \omega_{n-1}+\omega_n - E & -\omega_n & 0 \\ 0 & \omega_{n-1} & \omega_n - \omega_{n-1} & -\omega_n & 0 \\ 0 & 0 & -\omega_n & \omega_n + \omega_{n+1} - E & -\omega_{n+1} \\ 0 & 0 & 0 & \frac{E}{E'} & 0 \\ -\omega_{n-2} & \omega_{n-2}+\omega_{n-1} - E & -\omega_{n-1} & 0 & 0 \\ 0 & -\frac{E}{E'} & 0 & 0 & 0 \end{pmatrix},$$

$$A_1^2 := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -E' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Again, by using Lemma 4.2.5, we infer that

$$|\det A_1| \leqslant M_1 e^{-L^\beta/4}$$

where $M_1 = M_1(E, E', \alpha_0, \beta_0) > 0$.
Compute the determinant of A_1 (See Appendix A), we obtain

$$|\det A_1| = \frac{4E}{E'}\omega_{n-2}\omega_{n+1}\left|\omega_{n-1}\omega_n - \frac{(E-E')^2}{4}\right| \leqslant M_1 e^{-L^\beta/4}.$$

Hence, take $\omega_j \geqslant \alpha_0 > 0$ and $E, E' > 0$ into account, we have

$$\left|\omega_{n-1}\omega_n - \frac{(E-E')^2}{4}\right| \leqslant Ce^{-L^\beta/4} \tag{II}$$

as L sufficiently large.
Third case: $n-2, n-1 \in \mathcal{Q}_j$, $n \in \mathcal{P}_j$ and $n+1 \in \mathcal{Q}_{j+1}$.

According to Lemma 4.2.8, we have

$$A_2 U = b_2$$

where $\|b_2\| \leq e^{-L^\beta/2}$ and $A_2 = (A_2^1|A_2^2)$ is the 10×10 matrix defined by

$$A_2^1 := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -\omega_{n-1} & \omega_{n-1}+\omega_n - E & -\omega_n & 0 \\ 0 & \omega_{n-1} & \omega_n - \omega_{n-1} & -\omega_n & 0 \\ 0 & 0 & -\omega_n & \omega_n + \omega_{n+1} - E & -\omega_{n+1} \\ 0 & 0 & -\omega_n & \omega_n - \omega_{n+1} & \omega_{n+1} \\ -\omega_{n-2} & \omega_{n-2}+\omega_{n-1} - E & -\omega_{n-1} & 0 & 0 \\ 0 & -\dfrac{E}{E'} & 0 & 0 & 0 \end{pmatrix}$$

and

$$A_2^2 := \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -E' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E' & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 4.2.5 implies that $|\det A_2| \leq M_2 e^{-L^\beta/4}$ for some $M_2 > 0$.
Then, by an explicit computation, we obtain

$$|\det A_2| = 4E|E-E'|\omega_{n-2}\,\omega_{n+1}\left|\omega_n - \frac{E'+E}{4}\right| \leqslant M_2 e^{-L^\beta/4}$$

which yields that

$$\left|\omega_n - \frac{E'+E}{4}\right| \leqslant C e^{-L^\beta/4} \tag{III}$$

as $L \to +\infty$.

Forth case: Suppose that $n-2 \in \mathcal{Q}_j,\ n-1 \in \mathcal{P}_j,\ n \in \mathcal{Q}_{j+1}$ and $n+1 \in \mathcal{P}_{j+1}$.

In this case, U satisfies the following system of linear equations:

$$A_3 U = b_3$$

where $\|b_3\| \leq e^{-L^\beta/2}$ and $A_3 = (A_3^1|A_3^2)$ is the 10×10 matrix defined by

$$A_3^1 := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -\omega_{n-1} & \omega_{n-1}+\omega_n - E & -\omega_n & 0 \\ 0 & -\omega_{n-1} & \omega_{n-1}-\omega_n & \omega_n & 0 \\ 0 & 0 & -\omega_n & \omega_n+\omega_{n+1}-E & -\omega_{n+1} \\ 0 & 0 & \omega_n & \omega_{n+1}-\omega_n & -\omega_{n+1} \\ -\omega_{n-2} & \omega_{n-2}+\omega_{n-1}-E & -\omega_{n-1} & 0 & 0 \\ \omega_{n-2} & \omega_{n-1}-\omega_{n-2} & -\omega_{n-1} & 0 & 0 \end{pmatrix}$$

and

$$A_3^2 := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -E' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E' & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -E' & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, $|\det A_3| \leq M_3 e^{-L^\beta/4}$ thanks to Lemma 4.2.5.
We compute

$$|\det A_3| = EE'\omega_{n-2} \times \omega_{n+1} \times \Big|4\omega_{n-1} - (E'+E)\Big| \times \Big|4\omega_n - (E+E')\Big|.$$

Hence, at least one of the two following conditions on ω must be satisfied:

- $\left|\omega_{n-1} - \dfrac{E'+E}{4}\right| \leqslant Ce^{-L^\beta/8}, \quad (IV)$
- $\left|\omega_n - \dfrac{E'+E}{4}\right| \leqslant Ce^{-L^\beta/8}.$

From $(I)-(IV)$, Lemma 4.2.11 follows. □

Lemmata 4.2.10 and 4.2.11 yield that if we consider any 4 consecutive interior points of J, we obtain at least one condition of the types $(i)-(iii)$. Consequently, the random variables $\{\omega_j\}_{j\in\Lambda}$ must satisfy at least $|J|/8 = cL^\beta$ conditions of the types $(i)-(iii)$. From the fact that ω_n are i.i.d. and possess a bounded density, the conditions $(i)-(iii)$ imply that the event (*) defined in Lemma 4.2.3 can occur for a given partition $\mathcal{P}$ and $\mathcal{Q}$ with a probability at most $e^{-cL^{2\beta}}$ for some $c > 0$. Hence,

$$\mathbb{P}^* \leqslant 2^L e^{-cL^{2\beta}} \leqslant e^{-\widetilde{c}L^{2\beta}} \text{ with } 0 < \widetilde{c} < c$$

as the number of partitions is bounded by 2^L and $\beta > 1/2$.
We thus have Lemma 4.2.3 proved. □

Remark 4.2.12. *Thanks to the equality (4.2.3), it is not hard to derive the following estimate for the model (4.1.1),*

$$\frac{\Delta E}{2\beta_0}|\Lambda|^{-1/2} \leqslant \|\nabla_\omega(E(\omega) - E'(\omega))\| \leqslant \|\nabla_\omega(E(\omega) - E'(\omega))\|_1 \tag{4.2.33}$$

provided that $|E(\omega) - E| + |E'(\omega) - E'| \leqslant e^{-L^\beta}$ *and* $\Delta E = |E - E'|$.
The above estimate reads that the l^1-*distance of the gradients of* $E(\omega)$ *and* $E'(\omega)$ *is bounded from below by a positive term that is polynomially small w.r.t. the length of the interval* Λ.
Now, let $c_1 = c_2$ *in Lemma 4.2.3.*
Under the hypotheses in Lemma 4.2.3, the estimate (4.2.33) implies that, for any $\{\omega_j\}_{j\in\Lambda}$ *belonging to the event (*), we have*

$$\frac{\Delta E}{2\beta_0}|\Lambda|^{-1/2} \leqslant Ce^{-c|\Lambda|^\beta}$$

which is impossible when $|\Lambda|$ *sufficiently large. Hence, for* $c_1 = c_2$, $\mathbb{P}^*$ *is equal to* 0 .
Finally, we would like to note that an estimate like (4.2.33) for the discrete Anderson model holds true for two distinct energies sufficiently far apart from each other. This kind of estimate enable us to prove the decorrelation estimate for the discrete Anderson model in any *dimension (c.f. Lemma 2.4 in [Klo11]). But it is not the case for the model (4.1.1).*

4.3 Comment on the lower bound of the r.v.'s

In this section, we want to discuss how to relax the hypothesis on the lower bound of random variables $\{\omega_j\}_{j\in\mathbb{Z}}$.
Assume that all r.v.'s ω_j are only non-negative instead of being bounded from below by a positive constant. Precisely, assume that $\omega_j \in [0, \beta_0]$ $\forall j \in \mathbb{Z}$. In order to carry out our proof, we have to assume an extra condition on the distribution function $F(t)$ of random variables $\{\omega_j\}_j$: for some $\eta > 0$, we have

$$F(t) := \mathbb{P}(\omega_j \leqslant t) \leqslant e^{-t^{-\eta}} \tag{4.3.1}$$

for all small positive t, where η is some positive number.
The condition (4.3.1) means that the distribution $F(t)$ is exponentially small in a neighborhood of 0.
Now, let $\Lambda = [-L, L]$ be an interval in $\mathbb{Z}$, we have

$$\mathbb{P}(\exists \omega_\gamma \leqslant e^{-(\log L)^\delta} \text{ with } \gamma \in \Lambda) \leqslant (2L+1)e^{-\eta e^{(\log L)^\delta}} \tag{4.3.2}$$

where δ is a fixed number in $(0,1)$. Note that the right hand side of (4.3.2) converges to 0 as $L \to \infty$.
Hence, with a probability greater than or equal to $1 - (2L+1)e^{-\eta e^{(\log L)^\delta}}$,
we have

$$\omega_j \geq e^{-(\log L)^\delta} > 0 \; \forall j \in [-L, L]. \tag{4.3.3}$$

We will use (7.1.11) to prove the following "lower bound" for normalized eigenvectors of $H_\omega(\Lambda)$.

Lemma 4.3.1. *Pick $\beta \in (1/2, 1)$ and a fixed number $\epsilon \in (0, \beta)$. Let $\Lambda = [-L, L]$ be a large cube in $\mathbb{Z}$. Suppose that $E(\omega)$ is an eigenvalue of $H_\omega(\Lambda)$ and $u := u(\omega)$ is its associated normalized eigenvector.*
Then, with a probability greater than or equal to $1 - (2L+1)e^{-e^{(\log L)^\delta}}$,
there exists a point k_0 in Λ such that

$$u^2(k) + u^2(k+1) \geqslant e^{-L^\beta/2}$$

for all $|k - k_0| \leqslant \frac{1}{4}L^{\beta-\epsilon}$ as L large enough.

Proof of Lemma 4.3.1. Consider $\{\omega_j\}_{j\in\Lambda}$ such that (4.3.3) holds true. Using the same notations and proceed as in Lemma 4.2.2, for $n, m \in \Lambda$, we have

$$v(n) = T(n, E(\omega)) \cdots T(n-m+1, E(\omega))v(m)$$

where $T(n, E(\omega))$ and $v(n)$ are the transfer matrix and column vector defined in the proof of Lemma 4.2.2. Thanks to (4.3.3), the transfer matrices $T(n, E(\omega))$ are well defined and invertible. Moreover, they and their inverse matrices are bounded by $C_L := e^{c(\log L)^\delta}$ where $c > 0$ depends only on β_0.
Thus,

$$\|v(n)\| \leqslant C_L^{|n-m|}\|v(m)\| = e^{c(\log L)^\delta |n-m|}\|v(m)\| \tag{4.3.4}$$

where $n, m \in \Lambda$.
Assume that $\|v(k_0)\|$ is the maximum of $\|v(n)\|$. Hence,

$$\|v(k_0)\| \geq \frac{1}{\sqrt{2L}} \tag{4.3.5}$$

as u is a normalized vector.
Pick $\kappa > 0$ a fixed number and consider integers k such that $|k-k_0| \leqslant \kappa L^{\beta-\epsilon}$. From (4.3.4) and (4.3.5), we have the following inequality

$$\|v(k)\| \geqslant \frac{1}{\sqrt{2L}} e^{-c(\log L)^\delta |k-k_0|} \geq \frac{1}{\sqrt{2L}} e^{-c\kappa(\log L)^\delta L^{\beta-\epsilon}} \geq e^{-\kappa L^\beta}$$

when L is sufficiently large.
Hence, by choosing $\kappa = 1/4$, we have

$$u^2(k) + u^2(k+1) \geqslant e^{-L^\beta/2}$$

which completes the Lemma 4.3.1. □

Roughly speaking, we obtained almost the same "lower bound" for the normalized eigenvectors of finite volume operators, but with a good probability instead of the probability 1 as in Lemma 4.2.2.
Now, let β be a fixed number in the interval $(1/2, 1)$.
One the one hand, thanks to Lemma 4.3.1, the argument in proof of Theorem 4.2.3 still works out. In deed, in this case, we can proceed as in the proof of Lemma 4.2.3 to obtain at least $cL^{\beta-\epsilon}$ restrictions on r.v.'s ω. Hence, the upper bound for the probability $\mathbb{P}^*$ in Lemma 4.2.3 is now

$$B := 2^L (e^{-\tilde{c}L^\beta})^{cL^{\beta-\epsilon}} = 2^L e^{-cL^{2\beta-\epsilon}}.$$

If we choose ϵ in Lemma 4.3.1 small enough such that $2\beta - \epsilon > 1$, the upper bound B is exponentially small w.r.t. L. Hence, the Lemma 4.2.3 still holds true.
On the other hand, we observe that the upper bound of the probability that (4.3.3) fails, the term $(2L+1)e^{-\eta e^{(\log L)^\delta}} = o\left(\frac{l}{L}\right)$ as L large.
Hence, we obtain again the decorrelation estimate. In other words, Theorem 4.1.2 and Theorem 4.1.3 still hold true in this case.

4.4 More than two distinct energies

In this section, we would like to show that, following an argument in [Klo11], we can use Theorem 4.1.2 to prove the asymptotic independence for any fixed number of point processes.

Theorem 4.4.1. *For a fixed number $n \geqslant 2$, consider a finite sequence of fixed, positive energies $\{E_i\}_{1 \leqslant i \leqslant n}$ in the localized regime such that $\nu(E_i) > 0$ for all $1 \leq i \leq n$.*
Then, as $|\Lambda| \to +\infty$, n point processes $\Xi(\xi, E_i, \omega, \Lambda)$ defined as in (4.1.3) converge weakly to n independent Poisson processes.

Proof of Theorem 4.4.1. We will prove in detail the case of $n = 3$ with three distinct, positive energies E, E', E''.
Consider non-empty compact intervals $(U_j)_{1 \leqslant j \leqslant J}$, $(U'_j)_{1 \leqslant j \leqslant J'}$,$(U''_j)_{1 \leqslant j \leqslant J''}$ in $\mathbb{R}$ and integers $(k_j)_{1 \leqslant j \leqslant J}$, $(k'_j)_{1 \leqslant j \leqslant J'}$, $(k''_j)_{1 \leqslant j \leq J''}$ as in Theorem 4.1.3 i.e. for $k \neq j$, $U_j \cap U_k = \emptyset$, $U'_j \cap U'_k = \emptyset$ and $U''_j \cap U''_k = \emptyset$.
Using notations in [Klo11], one picks L and l such that $(2L+1) = (2l+1)(2l'+1)$ where $cL^\alpha \leqslant l \leqslant L^\alpha / c$ for some $\alpha \in (0,1)$ and $c > 0$. Then, one decomposes

$$\Lambda := [-L, L] = \bigcup_{|\gamma| \leqslant l'} \Lambda_l(\gamma)$$

where $\Lambda_l(\gamma) := (2l+1)\gamma + \Lambda_l$.
Next, for $\Lambda' \subset \Lambda$, $U \subset \mathbb{R}$ and $E > 0$, one defines the following Bernoulli r.v.

$$X(E, U, \Lambda') := \begin{cases} 1 & \text{if } H_\omega(\Lambda') \text{ has at least one eigenvalue} \\ & \text{in } E + (\nu(E)|\Lambda|)^{-1} U, \\ 0 & \text{otherwise} \end{cases} \tag{4.4.1}$$

and put $\Sigma(E, U, l) := \sum_{|\gamma| \leqslant l'} X(E, U, \Lambda_l(\gamma))$ where $(2l+1)(2l'+1) = (2L+1)$ with $cL^\alpha \leqslant l \leqslant L^\alpha / c$.
First of all, [Lemma 3.2, [Klo11]] is the first ingredient of the proof which tell us that we can actually reduce our problem to consider eigenvalues of finite-volume operators restricted on much smaller intervals. This lemma is still true in the n-energy case for all $n \geq 2$.
Then, to complete the proof of the stochastic independence w.r.t. three processes, one only need show that the quantity

$$\mathbb{P}\left(\left\{\omega; \begin{array}{cc} \Sigma(E, U_1, l) = k_1, & \ldots, \Sigma(E, U_J, l) = k_J \\ \Sigma(E', U'_1, l) = k'_1, & \ldots, \Sigma(E', U'_{J'}, l) = k'_{J'} \\ \Sigma(E", U''_1, l) = k''_1, & \ldots, \Sigma(E'', U''_{J''}, l) = k''_{J''} \end{array}\right\}\right) \tag{4.4.2}$$

should be approximated by the product

$$\mathbb{P}\left(\left\{\omega;\begin{array}{rr}\Sigma(E,U_1,l)= & k_1\\ \vdots & \vdots\\ \Sigma(E,U_J,l)= & k_J\end{array}\right\}\right)\times\mathbb{P}\left(\left\{\omega;\begin{array}{rr}\Sigma(E',U_1',l)= & k_1'\\ \vdots & \vdots\\ \Sigma(E',U_{J'}',l)= & k_{J'}'\end{array}\right\}\right)$$
$$\times\,\mathbb{P}\left(\left\{\omega;\begin{array}{rr}\Sigma(E'',U_1'',l)= & k_1''\\ \vdots & \vdots\\ \Sigma(E'',U_{J''}'',l)= & k_{J''}''\end{array}\right\}\right)$$

as L goes to infinity. Indeed, if the above statement is proved, Theorem 4.1.1 and Lemma 3.2 in [Klo11] yield immediately Theorem 4.4.1.

By a standard criterion of the convergence of point processes (c.f. e.g. Theorem 11.1.VIII, [DVJ08]), the above statement holds true if the following quantity vanishes for all real numbers $t_j, t'_{j'}, t''_{j''}$ when L goes to infinity:

$$\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\Sigma(E,U_j,l)-\sum_{j'=1}^{J'}t'_{j'}\Sigma(E',U'_{j'},l)-\sum_{j''=1}^{J''}t''_{j''}\Sigma(E'',U''_{j''},l)}\right)-$$
$$\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\Sigma(E,U_j,l)}\right)\mathbb{E}\left(e^{-\sum_{j'=1}^{J'}t'_{j'}\Sigma(E',U'_{j'},l)}\right)\mathbb{E}\left(e^{-\sum_{j''=1}^{J''}t''_{j''}\Sigma(E'',U''_{j''},l)}\right).$$

From the fact that $\{\Lambda(\gamma)\}_{|\gamma|\leqslant l'}$ are pairwise disjoint intervals, operators $\{H_\omega(\Lambda(\gamma))\}_{|\gamma|\leqslant l'}$ are independent operator-valued r.v.'s. We thus have

$$\begin{aligned}&\mathbb{E}\left(e^{-\sum_{j=1}^J t_j\Sigma(E,U_j,l)-\sum_{j'=1}^{J'}t'_{j'}\Sigma(E',U'_{j'},l)-\sum_{j''=1}^{J''}t''_{j''}\Sigma(E'',U''_{j''},l)}\right)\\&=\mathbb{E}\left|\prod_{|\gamma|\leqslant l'}e^{-\sum_j t_jX(E,U_j,\Lambda_l(\gamma))-\sum_{j'}t'_{j'}X(E',U'_{j'},\Lambda_l(\gamma))-\sum_{j''}t''_{j''}X(E'',U''_{j''},\Lambda_l(\gamma))}\right|\\&=\prod_{|\gamma|\leqslant l'}\mathbb{E}\left|e^{-\sum_j t_jX(E,U_j,\Lambda_l(\gamma))-\sum_{j'}t'_{j'}X(E',U'_{j'},\Lambda_l(\gamma))-\sum_{j''}t''_{j''}X(E'',U''_{j''},\Lambda_l(\gamma))}\right|.\end{aligned}$$

Our goal is to approximate terms of the form

$$\mathbb{E}\left(e^{-\sum_j^J t_jX(E,U_j,\Lambda_l(\gamma))-\sum_{j'}^{J'}t'_{j'}X(E',U'_{j'},\Lambda_l(\gamma))-\sum_{j''}^{J''}t''_{j''}X(E'',U''_{j''},\Lambda_l(\gamma))}\right)$$

by the product

$$\prod_j^J\mathbb{E}e^{-t_jX(E,U_j,\Lambda_l(\gamma))}\prod_{j'}^{J'}\mathbb{E}e^{-t'_{j'}X(E',U'_{j'},\Lambda_l(\gamma))}\prod_{j''}^{J''}\mathbb{E}e^{-t''_{j''}X(E'',U''_{j''},\Lambda_l(\gamma))}$$

as L large enough with the remark that these r.v.'s are not independent.
To illustrate our computation for that, we will consider just three arbitrary Bernoulli r.v.'s $X_i := X(E_i, U_i, \Lambda'),\ i \in \{1, 2, 3\}$ and compute explicitly $\mathbb{E}(e^{\sum_{i=1}^{3} a_i X_i})$.

Lemma 4.4.2. *Let E_1, E_2, E_3 be three positive energies in the localized regime. Pick three non-empty compact intervals $\{U_i\}_{i=\overline{1,3}}$ in $\mathbb{R}$ such that if $E_i = E_j$ with $i \neq j$, U_i and U_j are chosen to be disjoint sets. Assume that Λ' is a sub-interval of $\Lambda = [-L, L]$ such that $|\Lambda'| = 2l + 1 = O(L^\alpha)$ for some $\alpha \in (0, 1)$. Consider three Bernoulli r.v.'s $X_i := X(E_i, U_i, \Lambda')$ defined as in (4.4.1), we have*

$$\mathbb{E}(e^{\sum_{i=1}^{3} a_i X_i}) = \prod_{i=1}^{3} \mathbb{E}(e^{a_i X_i}) + O\left((l/L)^{1+\theta}\right) \tag{4.4.3}$$

for any $\theta \in (0, 1)$ as L large enough .

Proof of Lemma 4.4.2. Put $A := \mathbb{E}(e^{\sum_{i=1}^{3} a_i X_i})$, we have

$$A = \mathbb{P}(X_1 = X_2 = X_3 = 0) + \sum_{i=1}^{3} e^{a_i} \mathbb{P}\begin{pmatrix} X_i = 1, \\ X_j = 0\ \forall j \neq i \end{pmatrix}$$
$$+ \sum_{i<j} e^{a_i + a_j} \mathbb{P}\begin{pmatrix} X_i = X_j = 1, \\ X_k = 0; k \neq i, j \end{pmatrix} + e^{\sum_{i=1}^{3} a_i} \mathbb{P}(\bigcap_{i=1}^{3} \{X_i = 1\}).$$

First of all, we rewrite

$$\mathbb{P}(X_1 = X_2 = X_3 = 0) = 1 - \sum_{i=1}^{3} \mathbb{P}\begin{pmatrix} X_i = 1, \\ X_j = 0\ \forall j \neq i \end{pmatrix}$$
$$- \sum_{i<j} \mathbb{P}\begin{pmatrix} X_i = X_j = 1, \\ X_k = 0; k \neq i, j \end{pmatrix} - \mathbb{P}(\bigcap_{i=1}^{3} \{X_i = 1\})$$

and obtain that

$$A = 1 + \sum_{i=1}^{3} (e^{a_i} - 1) \mathbb{P}\begin{pmatrix} X_i = 1, \\ X_j = 0\ \forall j \neq i \end{pmatrix}$$
$$+ \sum_{i<j} (e^{a_i + a_j} - 1) \mathbb{P}\begin{pmatrix} X_i = X_j = 1, \\ X_k = 0; k \neq i, j \end{pmatrix} + (e^{\sum_{i=1}^{3} a_i} - 1) \mathbb{P}(\bigcap_{i=1}^{3} \{X_i = 1\}).$$

Next, use

$$\mathbb{P}\begin{pmatrix} X_i = 1, \\ X_j = 0 \;\forall j \neq i \end{pmatrix} = \mathbb{P}(X_i = 1) - \mathbb{P}\begin{pmatrix} X_i = X_{i+1} = 1, \\ X_{i+2} = 0 \end{pmatrix}$$
$$- \mathbb{P}\begin{pmatrix} X_i = X_{i-1} = 1, \\ X_{i+1} = 0 \end{pmatrix} - \mathbb{P}(\bigcap_{i=1}^{3}\{X_i = 1\})$$

to get

$$A = 1 + \sum_{i=1}^{3}(e^{a_i} - 1)\mathbb{P}(X_i = 1) + \sum_{i<j}(e^{a_i} - 1)(e^{a_j} - 1)\mathbb{P}\begin{pmatrix} X_i = X_j = 1, \\ X_k = 0 \; k \neq i, j \end{pmatrix}$$
$$+ \left(e^{\sum_{i=1}^{3} a_i} - 1 - \sum_{i=1}^{3}(e^{a_i} - 1)\right)\mathbb{P}(\bigcap_{i=1}^{3}\{X_i = 1\}).$$

Using a similar expansion for all terms of the form $\mathbb{P}\begin{pmatrix} X_i = X_j = 1, \\ X_k = 0 \; k \neq i, j \end{pmatrix}$, we obtain the following formula

$$A = 1 + \sum_{i=1}^{3}(e^{a_i} - 1)\mathbb{P}(X_i = 1) + \sum_{i<j}(e^{a_i} - 1)(e^{a_j} - 1)\mathbb{P}\left(X_i = X_j = 1\right)$$
$$+ \prod_{i=1}^{3}(e^{a_i} - 1)\mathbb{P}(\bigcap_{i=1}^{3}\{X_i = 1\}). \tag{4.4.4}$$

On the other hand, from the observation that

$$\mathbb{E}e^{a_j X_j} = 1 + (e^{a_j} - 1)\mathbb{P}(X_j = 1) \;\forall j = \{1, 2, 3\},$$

we multiply the three equalities above to get

$$\prod_{i=1}^{3}\mathbb{E}(e^{a_i X_i}) = 1 + \sum_{i=1}^{3}(e^{a_i} - 1)\mathbb{P}(X_i = 1)$$
$$+ \sum_{i<j}(e^{a_i} - 1)(e^{a_j} - 1)\mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1) + \prod_{i=1}^{3}(e^{a_i} - 1)\mathbb{P}(X_i = 1). \tag{4.4.5}$$

Hence, thanks to (4.4.4) and (4.4.5), we have

$$\begin{aligned}
&\mathbb{E}\left(e^{\sum_{i=1}^{3} a_i X_i}\right) - \prod_{i=1}^{3} \mathbb{E}(e^{a_i X_i}) = \\
&\sum_{i<j} (e^{a_i} - 1)(e^{a_j} - 1)\left[\mathbb{P}(X_i = X_j = 1) - \mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1)\right] \qquad (4.4.6)\\
&+ \prod_{i=1}^{3} (e^{a_i} - 1)\left[\mathbb{P}(\bigcap_{j=1}^{3}\{X_j = 1\}) - \prod_{j=1}^{3} \mathbb{P}(X_j = 1)\right].
\end{aligned}$$

Theorem 4.1.2 and Theorem 3.1.1 yield, for any $\theta \in (0,1)$,

$$\begin{aligned}
&\mathbb{P}(X_i = X_j = 1) + \mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1)\\
&\leqslant C(l/L)^2(e^{(\log L)^\beta} + 1) \leqslant C(l/L)^{1+\theta} \qquad (4.4.7)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}(\bigcap_{i=1}^{3}\{X_i = 1\}) + \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1)\mathbb{P}(X_3 = 1)\\
&\leqslant C(l/L)^2(e^{(\log L)^\beta} + (l/L)) \leqslant C(l/L)^{1+\theta} \qquad (4.4.8)
\end{aligned}$$

with L large enough. Note that C is a positive constant depending only on $\{E_i\}_{i=1}^3$ and $\{U_i\}_{i=1}^3$.
From (4.4.6)-(4.4.8), Lemma 4.4.2 follows. □

It is easy to see that the computation in Lemma 4.4.2 can apply to any finite number of Bernoulli r.v.'s satisfying the hypotheses in Lemma 4.4.2. Hence, for each $|\gamma| \leqslant l'$, we have

$$\begin{aligned}
&\mathbb{E}\left(e^{-\sum_j t_j X(E,U_j,\Lambda_l(\gamma)) - \sum_{j'} t'_{j'} X(E',U'_{j'},\Lambda_l(\gamma)) - \sum_{j''} t''_{j''} X(E'',U''_{j''},\Lambda_l(\gamma))}\right)\\
&= \prod_{j}^{J} \mathbb{E}e^{-t_j X(E,U_j,\Lambda_l(\gamma))} \prod_{j'}^{J'} \mathbb{E}e^{-t'_{j'} X(E',U'_{j'},\Lambda_l(\gamma))} \prod_{j''}^{J''} \mathbb{E}e^{-t''_{j''} X(E'',U''_{j''},\Lambda_l(\gamma))}\\
&\times (1 + O(l/L)^{1+\theta}).
\end{aligned}$$

On the other hand, we also have similar formulas for $\mathbb{E}e^{-\sum_j t_j X(E,U_j,\Lambda_l(\gamma))}$, $\mathbb{E}e^{-\sum_{j'} t'_{j'} X(E',U'_j,\Lambda_l(\gamma))}$ and $\mathbb{E}e^{-\sum_{j''} t''_j X(E'',U''_{j''},\Lambda_l(\gamma))}$.
We thus have

$$\begin{aligned}
&\mathbb{E}\left(e^{-\sum_j t_j X(E,U_j,\Lambda_l(\gamma)) - \sum_{j'} t'_{j'} X(E',U'_{j'},\Lambda_l(\gamma)) - \sum_{j''} t''_{j''} X(E'',U''_{j''},\Lambda_l(\gamma))}\right)\\
&= \mathbb{E}e^{-\sum_j t_j X(E,U_j,\Lambda_l(\gamma))}\mathbb{E}e^{-\sum_{j'} t'_{j'} X(E',U'_{j'},\Lambda_l(\gamma))}\mathbb{E}e^{-\sum_{j''} t''_{j''} X(E'',U''_{j''},\Lambda_l(\gamma))}\\
&\times (1 + O(l/L)^{1+\theta}).
\end{aligned}$$

We have an observation that $|\gamma| \leqslant l'$ where $l' = O(L/l)$. Hence, by multiplying all the above equalities side by side over $|\gamma| \leq l'$, we obtain that:

$$\mathbb{E}\left(e^{-\sum_{j=1}^{J} t_j \Sigma(E,U_j,l) - \sum_{j'=1}^{J'} t'_{j'} \Sigma(E',U_{j'},l) - \sum_{j''=1}^{J''} t''_{j''} \Sigma(E'',U_{j''},l)}\right)$$

is equal to the product of

$$\mathbb{E}\left(e^{-\sum_{j=1}^{J} t_j \Sigma(E,U_j,l)}\right) \mathbb{E}\left(e^{-\sum_{j'=1}^{J'} t'_{j'} \Sigma(E',U_{j'},l)}\right) \mathbb{E}\left(e^{-\sum_{j''=1}^{J''} t''_{j''} \Sigma(E'',U_{j''},l)}\right)$$

and an error term of the form $(1 + x^{1+\theta})^{1/x}$ with $x = O(l/L)$.
Note that the above error term tends to 1 as L goes to infinity. Hence, the stochastic independence for three point processes w.r.t. three positive, distinct energies is proved. Finally, it is not hard to see that we can adapt this proof for $n-$energy case with any $n \geq 2$. □

4.5 An alternative proof of decorrelation estimates for 1D discrete Anderson model

In dimension 1, the discrete Anderson model can be defined as follows: for $u \in l^2(\mathbb{Z})$, set

$$(H^A_\omega u)(n) = u(n-1) + u(n+1) + \omega_n u(n) \tag{4.5.1}$$

where $\omega := \{\omega_n\}_{n\in\mathbb{Z}}$ are i.i.d. random variables (r.v.'s for short) with a bounded, compactly supported density ρ.
As mentioned in Section 4.1, the decorrelation estimate for the 1D discrete Anderson model was settled in [Klo11] and that proof can not be applied to the lattice Hamiltonian with off-diagonal disorder, the model (2.1.3). So, a new approach for proving decorrelation estimates for (2.1.3) is necessary and its essence is presented in Lemma 4.2.3 of Section 4.2.
Does the above-mentioned approach also works on the 1D discrete Anderson model?
To answer this question, it suffices to check if the proof of Lemma 4.2.3 with $c_1 = c_2 = 1$ still holds or not for the 1D discrete Anderson model. Fortunately, the answer is affirmative:

Lemma 4.5.1. *Let $E \neq E'$ be two positive energies in the localized regime and $\beta \in (1/2, 1)$. Assume that $\Lambda = \Lambda_L =: [-L, L]$ is a large interval in $\mathbb{Z}$. Denote by $\mathbb{P}^*$ the probability of the following event (called (*)):*

There exists two simple eigenvalues of $H_\omega(\Lambda)$*, say* $E(\omega), E'(\omega)$ *such that* $|E(\omega) - E| + |E'(\omega) - E'| \leqslant e^{-L^\beta}$ *and*

$$\|\nabla_\omega(E(\omega) - E'(\omega))\|_1 \leqslant e^{-L^\beta}.$$

Then, there exists $c > 0$ *such that*

$$\mathbb{P}^* \leqslant e^{-cL^{2\beta}}.$$

Proof of Lemma 4.5.1. Proceed as in the proof of Lemma 4.2.3, let $u := u(\omega)$ and $v := v(\omega)$ be normalized eigenvectors associated to eigenvalues $E(\omega)$ and $E'(\omega)$ of $H^A_\omega(\Lambda)$ with $\Lambda := [-L, L]$.

It is easy to check that

$$\partial_{\omega_n} E(\omega) = |u(n)|^2 \text{ for all } n \in \Lambda.$$

Under the assumption of Lemma 4.2.3 with $c_1 = c_2 = 1$, i.e.,

$$\|\nabla(E(\omega) - E'(\omega))\|_1 = \sum_{n\in\Lambda} |\partial_{\omega_n} E(\omega) - \partial_{\omega_n} E'(\omega)| \leq e^{-L^\beta},$$

there exists a decomposition of Λ into $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P} \cap \mathcal{Q} = \emptyset$ and

- for $n \in \mathcal{P}, |u(n) - v(n)| \leqslant e^{-L^\beta/2}$,
- for $n \in \mathcal{Q}$, $|u(n) + v(n)| \leqslant e^{-L^\beta/2}$.

Besides, when we use again transfer matrices associated to the 1D Anderson model, we obtain the same lower bound for normalized eigenvectors of $H^A_\omega(\Lambda)$ i.e., there exists a subinterval J of the length $O(L^\beta)$ of Λ such that

$$|u(k)|^2 + |u(k+1)|^2 \geqslant 2e^{-L^\beta/3} \tag{4.5.2}$$

for all $k \in J$. Then, we decompose

$$\mathcal{P} \cap J = \cup \mathcal{P}_j \text{ and } \mathcal{Q} \cap J = \cup \mathcal{Q}_j \tag{4.5.3}$$

where $\mathcal{P}_j$ and $\mathcal{Q}_j$ are intervals in $\mathbb{Z}$.

As in the case of (2.1.3), we can easily show a restriction on length of the intervals $\mathcal{P}_j$ and $\mathcal{Q}_j$. To be precise, any $\mathcal{P}_j$ or $\mathcal{P}_j$ can not contain more than 3 points.

Lemma 4.5.2. *Let* $E \neq E'$ *be two positive energies in the localized regime and* $\beta \in (1/2, 1)$*. Assume that* $\Lambda = \Lambda_L =: [-L, L]$ *is a large interval in* $\mathbb{Z}$*.*

Pick $c_1, c_2 > 0$ *and denote by* $\mathbb{P}^*$ *the probability of the following event (called (*)):*

There exists two simple eigenvalues of $H_\omega(\Lambda)$*, say* $E(\omega), E'(\omega)$ *such that* $|E(\omega) - E| + |E'(\omega) - E'| \leqslant e^{-L^\beta}$ *and*

$$\|\nabla_\omega(E(\omega) - E'(\omega))\|_1 \leqslant e^{-L^\beta}.$$

Denote by u, v normalized eigenvectors associated to $E(\omega), E'(\omega)$ respectively and consider the decomposition $\{\mathcal{P}_i, \mathcal{Q}_j\}$ in (4.5.3). Then, any $\mathcal{P}_j$ or $\mathcal{Q}_j$ can not contain more than three points.

Proof of Lemma 4.5.2. Because of the equivalent role of $\mathcal{P}$ and $\mathcal{Q}$, it suffices to prove the present lemma for $\{\mathcal{P}_j\}$. Assume by contradiction that there exist four consecutive points in J belonging to $\mathcal{P}_j$, say $n-2, n-1, n, n+1$. The lower bound (4.5.2) implies that

$$|u(n-1)|^2 + |u(n)|^2 \geqslant 2e^{-L^\beta/3}.$$

W.o.l.g., we assume that $|u(n)| \geq e^{-L^\beta/6}$. Let's consider three points $n-1, n, n+1$. Then, eigenequations for u and v at n read that

$$u(n-1) + u(n+1) + (\omega_n - E)u(n) = O(e^{-L^\beta/2}), \tag{4.5.4}$$

$$v(n-1) + v(n+1) + (\omega_n - E')v(n) = O(e^{-L^\beta/2}). \tag{4.5.5}$$

Since $\{n-1, n, n+1\}$ belongs to $\mathcal{P}_j$, (4.5.5) can be rewritten as

$$u(n-1) + u(n+1) + (\omega_n - E')u(n) = O(e^{-L^\beta/2}). \tag{4.5.6}$$

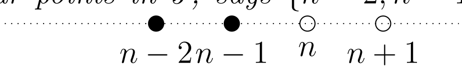
Hence, (4.5.6) and (4.5.4) yield that

$$|E - E'||u(n)| \leq Ce^{-L^\beta/2}. \tag{4.5.7}$$

However, when L large enough, the above inequality contradicts the assumption that $|u(n)| \geq e^{-L^\beta/6}$. Hence, any $\mathcal{P}_j$ or $\mathcal{Q}_j$ can not contain more than three points. □

Remark 4.5.3. *From (4.5.7), we observe that the situation where three points in the same interval $\mathcal{P}_j$ and the absolute value of u at the middle point of these three points is not too small can not happen either. In other words, under some additional assumption on values of eigenvectors u, $\mathcal{P}_j$ even can not contain more than two points.*

Lemma 4.5.4. *Assume the same hypotheses in Lemma 4.5.2. In addition, assume that there exist four points in J, says $\{n-2, n-1, n, n+1\}$, such that $n-2, n-1 \in \mathcal{P}_j$, $n, n+1 \in \mathcal{Q}_j$*



Then, $|E - E'| = 2$ and we have at least one of following restrictions on random variables $\{\omega_j\}_{j \in \Lambda}$

(I) $[(\omega_n + 1 - E)(\omega_{n+1} - E + 1) - 1] \leq Ce^{-cL^\beta/100}$,

(II) $|\omega_n - E + 1| \leq Ce^{-cL^\beta/100}$.

Proof of Lemma 4.5.4. We write down the eigenequations of u and v at point $n-1$:

$$u(n-2) + u(n) + (\omega_{n-1} - E)u(n-1) = O(e^{-L^\beta/2}), \tag{4.5.8}$$

$$v(n-2) + v(n) + (\omega_{n-1} - E')v(n-1) = O(e^{-L^\beta/2}). \tag{4.5.9}$$

Since the hypotheses on $u(j)$ and $v(j)$, $j = \overline{n-2, n}$ and (4.5.9), we have

$$u(n-2) - u(n) + (\omega_{n-1} - E')u(n-1) = O(e^{-L^\beta/2}). \tag{4.5.10}$$

Combining (4.5.8) with (4.5.10), we infer that

$$u(n) + \frac{E'-E}{2}u(n-1) = O(e^{-L^\beta/2}). \tag{4.5.11}$$

Repeating the above argument for eigenequations of u, v at n, we obtain

$$u(n-1) + \frac{E'-E}{2}u(n) = O(e^{-L^\beta/2}). \tag{4.5.12}$$

Combining (4.5.11), (4.5.12) with the fact that $|u(n-1)|^2 + |u(n)|^2 \geq e^{-L^\beta/3}$, we infer that $|E - E'| = 2$. To deduce some restrictions on random variables $\{\omega_j\}_{j\in\Lambda}$, we follow the strategy in Lemma 4.2.8 to make a square system of linear equations from these four points.

W.l.o.g., let's assume that $E - E' = 2$.

First of all, we use (4.5.8), the eigenequation of u at $n-1$, and (4.5.11), the relation between $u(n)$ and $u(n-1)$, to form the first and second linear equation in our square system.

Then, we use (4.5.4), the eigenequation of u at n as the third linear equation.

Finally, consider the eigenequations of u and v at the point $n+1$ which can be written as

$$u(n) + u(n+2) + (\omega_{n+1} - E)u(n+1) = O(e^{-L^\beta/2}), \tag{4.5.13}$$

$$-u(n) \pm u(n+2) - (\omega_{n+1} - E')u(n+1) = O(e^{-L^\beta/2}). \tag{4.5.14}$$

To sum up, we obtain a 5×5 inhomogeneous system of linear equations

$$AU = b$$

where $U := (u(n-2), u(n-1), \dots, u(n+2))$, b is a small vector in norm and A is the following 5×5 matrix

$$A = (a_{ij})_{1 \le i,j \le 5} = \begin{pmatrix} 1 & \omega_{n-1} - E & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & \omega_n - E & 1 & 0 \\ 0 & 0 & 1 & \omega_{n+1} - E & 1 \\ 0 & 0 & -1 & E - \omega_{n+1} - 2 & \pm 1 \end{pmatrix}$$

Applying Lemma 4.2.5 to the system $AU = b$, we infer that $\det A$ is exponentially small. By an explicit computation, we obtain that
either

$$|\det A| = 2\,|(\omega_n - E + 1)(\omega_{n+1} - E + 1) - 1| \le Ce^{-cL^\beta/100} \tag{I}$$

when $a_{55} = 1$,
or

$$|\det A| = 2|\omega_n - E + 1| \le Ce^{-cL^\beta/100} \tag{II}$$

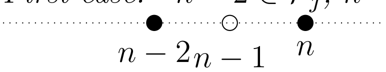
when $a_{55} = -1$. □

Now, let's consider four arbitrarily consecutive points in J, say $n-2, n-1, n, n+1$. The lower bound (4.5.2) yields that at least one of two numbers $|u(n-1)|$ and $|u(n)|$ must be greater than $e^{-L^\beta/6}$. Let's assume that $|u(n-1)|$ is the one and we will see what can be derived from three points $n-2, n-1, n$.

Lemma 4.5.5. *Assume the same hypotheses as in Lemma 4.5.2. Let $n-2, n-1, n$ be three points in J such that $|u(n-1)| \ge e^{-L^\beta/6}$.*
Then, random variables $\{\omega_j\}_{j\in\Lambda}$ satisfy at least one of conditions (I), (II) in Lemma 4.5.4 or the following condition

(III) $\left|\omega_{n-1} - \dfrac{E+E'}{2}\right| \le Ce^{-L^\beta/3}$.

Proof. Each point $n-2, n-1, n$ can belong to either $\mathcal{P}$ or $\mathcal{Q}$. So we have $2^3 = 8$ configurations for these points. By the equivalent role of $\mathcal{P}$ and $\mathcal{Q}$, we only need to consider following 4 configurations for these three points:
First case: $n-2 \in \mathcal{P}_j$, $n-1 \in \mathcal{Q}_j$, $n \in \mathcal{P}_j$ and $|u(n-1)| \ge e^{-L^\beta/6}$



Applying the eigenequation for u and v at $n-1$, we have

$$u(n-2) + u(n) + (\omega_{n-1} - E)u(n-1) = O(e^{-L^\beta/2}) \tag{4.5.15}$$

and

$$v(n-2) + v(n) + (\omega_{n-1} - E')v(n-1) = O(e^{-L^\beta/2}). \tag{4.5.16}$$

In the present case, (4.5.16) can be rewritten as

$$u(n-2) + u(n) - (\omega_{n-1} - E')u(n-1) = O(e^{-L^\beta/2}). \tag{4.5.17}$$

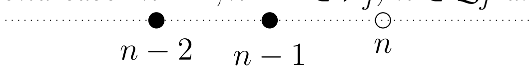
Substitute (4.5.17) into (4.5.15), we infer that

$$\left(\omega_{n-1} - \frac{E+E'}{2}\right) u(n-1) = O(e^{-L^\beta/2}).$$

Combining the above inequality and $|u(n-1)| \geq e^{-L^\beta/6}$, we deduce that

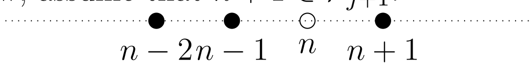
$$\left|\omega_{n-1} - \frac{E+E'}{2}\right| \leq Ce^{-L^\beta/3}. \tag{4.5.18}$$

Second case: $n-2, n-1 \in \mathcal{P}_j$, $n \in \mathcal{Q}_j$ *and* $|u(n-1)| \geq e^{-L^\beta/6}$



Let's take a look at the point $n+1$. If $n+1 \in \mathcal{Q}$, Lemma 4.5.4 implies the restrictions of types (I)-(II) for $(\omega_j)_j$.

Now, assume that $n+1 \in \mathcal{P}_{j+1}$.



In the present case, (4.5.8) and (4.5.9), the eigenequations of u, v at $n-1$, can be rewritten as

$$u(n-2) + u(n) + (\omega_n - E)u(n-1) = O(e^{-L^\beta/2}), \tag{4.5.19}$$

$$u(n-2) - u(n) + (\omega_n - E')u(n-1) = O(e^{-L^\beta/2}). \tag{4.5.20}$$

Take (4.5.19) minus (4.5.20) to get

$$u(n) + \frac{E'-E}{2}u(n-1) = O(e^{-L^\beta/2}). \tag{4.5.21}$$

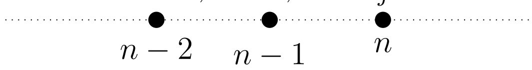
Similarly, using (4.5.15), (4.5.16) and eigenequations of u, v at n, we infer that

$$\left(\omega_n - \frac{E+E'}{2}\right) u(n) = O(e^{-L^\beta/2}). \tag{4.5.22}$$

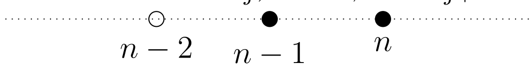
Substitute (4.5.22) into (4.5.21) and use the hypothesis $|u(n-1)| \geq e^{-L^\beta/6}$, we have

$$|E-E'|\left|\omega_n - \frac{E+E'}{2}\right| \leq Ce^{-L^\beta/3}. \tag{III}$$

Third case: $n-2, n-1, n \in \mathcal{P}_j$



Then, the point $n+1$ must belong to some $\mathcal{Q}_j$.
If $|u(n)| \geq e^{-L^\beta/6}$, we consider points $\{n-1, n, n+1\}$ and come back to *Second case*. Otherwise, $|u(n)| < e^{-L^\beta/6}$ which yields $|u(n+1)| \geq e^{-L^\beta/6}$. Let's consider the point $n+2$. If $n+2 \in \mathcal{P}_{j+1}$, we consider the points $\{n, n+1, n+2\}$ and come back to *First case*. Otherwise, $n+2 \in \mathcal{Q}_j$. Then, we can apply Lemma 4.5.4 to four points $\{n-1, n, n+1, n+2\}$ and obtain restrictions of types (I)-(II) for random variables.
Note that we need not use any conditions on values of u in this case.
Last case: $n-2 \in \mathcal{Q}_j$, $n-1, n \in \mathcal{P}_{j+1}$



Consider the point $n+1$. If $n+1 \in \mathcal{P}$, we consider the triple $\{n-1, n, n+1\}$ and come back to *Third case*. Otherwise, $n+1 \in \mathcal{Q}$. Hence, if $|u(n)| \geq e^{-L^\beta/6}$, we consider the triple $\{n-1, n, n+1\}$ and come back to *Second case*. Now, let's assume that $|u(n)| < e^{-L^\beta/6}$. Hence, $|u(n+1)| \geq e^{-L^\beta/6}$. Let's take a look at the point $n+2$. If $n+2 \in \mathcal{P}$, we consider $\{n, n+1, n+2\}$ and come back to *First case*. Otherwise, $n+2 \in \mathcal{Q}$ and Lemma 4.5.4 yields restrictions on random variables. □

From Lemmata 4.5.4 and 4.5.5, we see that if we consider any 6 consecutive points in J, we obtain at least one condition of types (I)-(III). Consequently, the random variables $\{\omega_j\}_{j\in\Lambda}$ must satisfy at least $|J|/6 = cL^\beta$ conditions of types (I)-(III). From the fact that ω_n are i.i.d. and possess a bounded density, (I)-(III) imply that the event (*) defined in Lemma 4.2.3 can occur for a given partition $\mathcal{P}$ and $\mathcal{Q}$ with a probability at most $e^{-cL^{2\beta}}$ for some $c > 0$. Hence,

$$\mathbb{P}^* \leqslant 2^L e^{-cL^{2\beta}} \leqslant e^{-\widetilde{c}L^{2\beta}} \text{ with } 0 < \widetilde{c} < c$$

as the number of partitions is bounded by 2^L and $\beta > 1/2$.
We thus have Lemma 4.5.1 proved. □

4.6 A little bit more about local level statistics

In Theorem 4.1.1 or 4.1.3, to get Poissonian statistics near an energy E, we always assume that the value of the density of states at this energy is positive. In other words, we study local level statistics in the bulk of spectrum. Recently, Germinet and Klopp prove in [GK11a] enhanced Wegner and Minami estimates in the strong dynamical localization regime Σ_{SDL} for discrete Anderson model which allows them to weaken the hypothesis $\nu(E) > 0$ and study the spectral statistics near edges of spectrum for the same operator in dimension 1.

The difference in their new Wegner and Minami type estimates is the replacement of the length of interval I by the quantity $N(I)$ as long as $N(I)$ is not too small:

Theorem 4.6.1. *[Theorem 2.1, [GK11a]] Fix $\xi \in (0,1)$. There exists constants c, C such that for $L > 1$ the following holds*

1. *Let $I \subset \Sigma_{SDL}$ be a compact interval. Then*

$$|\mathbb{E}tr1_I(H_\omega(\Lambda) - N(I)|\Lambda|)| \leq C\exp(-cL^\xi).$$

As a consequence, if $N(I) \geq C\exp(-cL^\xi)$, we get the Wegner estimate:

$$\mathbb{E}(tr1_I(H_\omega)) \leq 2N(I)|\Lambda|.$$

2. *If $N(I) \geq C\exp(-cL^\xi)$, we get the Minami estimate*

$$\mathbb{E}[tr1_I(H_\omega)(tr1_I(H_\omega) - 1)] \leq 2N(I)|I||\Lambda|^2.$$

Thanks to Theorem 4.6.1, for the d-dimension Anderson model, they can prove Poissonian statistics near an energy $E \in \Sigma_{\text{SDL}}$ under the assumption

$$|N(E+\varepsilon) - N(E)| \geq e^{-|\varepsilon|^{-\rho}} \text{for } \varepsilon \text{ small enough} \tag{4.6.1}$$

where $\rho \in (0, 1/d)$.

Note that the above assumption is satisfied whenever $\nu(E) > 0$.

It is not hard to figure out that, by proceeding as in the proof of [Theorem 2.1, [GK11a]], we can prove enhanced Wegner and Minami estimates for the model (4.1.1) wherever the "usual" Wegner holds i.e., we have enhanced Wegner and Minami estimates everywhere in the almost sure spectrum except at the energy 0.

On the other hand, in dimension 1, the Anderson localization appears everywhere in the spectrum of (4.1.1) and the strong dynamical localization regime Σ_{SDL} coincides with the almost sure spectrum.

Hence, we have the same Poissonian statistics and asymptotic independence as in Theorem 4.1.1 and Theorem 4.1.3 for (4.1.1) in dimension 1 but under a weaker assumption, the hypothesis (4.6.1).

However, not as for the discrete Anderson model in dimension 1, we can not obtain the spectral statistics at the bottom of spectrum 0 for the model (4.1.1) since the corresponding Wegner estimate does not hold at 0 (See Theorem 3.1.1, Chapter 3).

CHAPTER 5

RESONANCE STATISTICS IN DIMENSION 1-WHAT ARE ALREADY KNOWN

Contents

In this chapter, we would like to remind readers of some recent results obtained by [Klo]Klopp on resonances of a 1D discrete, truncated periodic Schrödinger operator.
Let V be a periodic potential of period p and $-\Delta$ be the (negative) discrete Laplacian on $l^2(\mathbb{Z})$. We define 1D Schrödinger operators $H^{\mathbb{Z}} := -\Delta + V$ acting on $l^2(\mathbb{Z})$:

$$(H^{\mathbb{Z}}u)(n) = ((-\Delta + V)u)(n) = u(n-1) + u(n+1) + V(n)u(n), \quad \forall n \in \mathbb{Z} \tag{5.0.1}$$

and $H^{\mathbb{N}} := -\Delta + V$ acting on $l^2(\mathbb{N})$ with Dirichlet boundary condition (b.c.) at 0.
Denote by $\Sigma_{\mathbb{Z}}$ the spectrum of $H^{\mathbb{Z}}$ and $\Sigma_{\mathbb{N}}$ the spectrum of $H^{\mathbb{N}}$. One has the following description for the spectra of $H^{\bullet}$ where $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$:

- $\Sigma_{\mathbb{Z}}$ is the union of disjoint intervals; the spectrum of $H^{\mathbb{Z}}$ is purely absolutely continuous (a.c.) and the spectral resolution can be obtained via the Bloch-Floquet decomposition (see [vM76] for more details).
- $\Sigma_{\mathbb{N}} = \Sigma_{\mathbb{Z}} \cup \{v_i\}_{i=1}^m$ where $\Sigma_{\mathbb{Z}}$ is the a.c. spectrum of $H^{\mathbb{N}}$ and $\{v_i\}_{i=1}^m$ are isolated simple eigenvalues of $H^{\mathbb{N}}$ associated to exponentially decaying eigenfunctions (c.f. [Pav94]).

Next, we introduce some auxiliary operators which will be used in next sections.
First of all, we define the translates of $H^{\mathbb{Z}}$ restricted to the negative axis: Pick some $j \geq 0$. On $l^2(\mathbb{Z}_-)$, where $\mathbb{Z}_- = \{n \leq 0\}$, we consider $H_j^- := -\Delta + \tau_j V$ with Dirichlet boundary condition at 0 where $\tau_j V(\cdot) = V(\cdot + j)$. It is well known that $\sigma_{\text{ess}}(H_j^-) = \Sigma_{\mathbb{Z}}$ and it is purely a.c. Moreover, H_j^- may have discrete eigenvalues in $\mathbb{R}\backslash\Sigma_{\mathbb{Z}}$ (c.f. [Tes00, Chapter 7]).
Next, we define the operator $H_0^+ := H^{\mathbb{N}} = -\Delta + V$ considered on $l^2(\mathbb{N})$ with Dirichlet boundary condition at 0. Its spectral properties are similar to those of H_j^-. Note that the number of eigenvalues of H_0^+ and H_j^- outside $\Sigma_{\mathbb{Z}}$ is finite.
Now, we pick a large natural number L and set:

$$H_L^{\mathbb{N}} := -\Delta + V\mathbb{1}_{[0,L]} \text{ on } l^2(\mathbb{N}) \text{ with Dirichlet boundary conditions (b.c.) at } 0.$$

Throughout the present chapter, L and j are always chosen such that $L \equiv j \mod (p)$ where p is the period of the potential V.
It is easy to check that the operator $H_L^{\mathbb{N}}$ is self-adjoint. Then, the resolvents $z \in \mathbb{C}^+ \mapsto (z - H_L^{\mathbb{N}})^{-1}$ are well defined on $\mathbb{C}^+$. Moreover, one can show that $z \mapsto (z - H_L^{\mathbb{N}})^{-1}$ admits a meromorphic continuation from $\mathbb{C}^+$ to $\mathbb{C}\backslash((-\infty, -2] \cup [2, +\infty))$ with values in the self-adjoint operators from l^2_{comp} to l^2_{loc}:

Theorem 5.0.2. *[Klo, Theorem 1.1] The resolvent $z \in \mathbb{C}^+ \mapsto (z - H_L^{\mathbb{N}})^{-1}$ admits a meromorphic continuation from $\mathbb{C}^+$ to $\mathbb{C} \setminus ((-\infty, -2] \cup [2, +\infty))$ with values in the operators from l^2_{comp} to l^2_{loc}.*
Moreover, the number of poles of the meromorphic continuation in the lower half-plane $\{ImE < 0\}$ is at most equal to L.

Now, we define the *(quantum) resonances* of $H_L^{\mathbb{N}}$, the main objet to study in the present chapter, as the poles of the above meromorphic continuation. The resonance widths, the imaginary parts of resonances, play an important role in the large time behavior of wave packets, especially the resonances of the smallest width that give the leading order contribution (see [SZ91] for an intensive study of resonances in the continuous setting and [IK12, IK14b, IK14a, BNW05, Kor11] for a study of resonances of various 1D operators). The distribution of resonances of $H_L^{\mathbb{N}}$ in the limit $L \to +\infty$ was studied intensively in [Klo]. All results proved in [Klo] assume that the real part of resonances are far from the boundary point of the spectrum $\Sigma_{\mathbb{Z}}$ and far from the point ± 2, the boundary of the essential spectrum of free Laplacian $-\Delta$. By "far", we mean the distance between resonances and $\partial\Sigma_{\mathbb{Z}}$ and ± 2 is bigger than a positive constant independent of L.
In this chapter, we make a quick summary on what were known about the resonances far from the boundary $\partial\Sigma_{\mathbb{Z}}$ of $\Sigma_{\mathbb{Z}}$ and points ± 2 in [Klo].
We will see in the chapters 6 and 7 that new phenomena will happen for resonances whose real parts are near "singular" points which are boundary points of $\Sigma_{\mathbb{Z}}$. Around these points, we choose different approaches to get the description of resonances.

5.1 Resonance equation

Theorem 5.1.1. *[Klo, Theorem 2.1] Consider the operator H_L defined as $H_L^{\mathbb{N}}$ restricted to $[0, L]$ with Dirichlet b.c. at L and define*

- *Denote by $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_L$ the Dirichlet eigenvalues of H_L;*
- *$a_l^{\mathbb{N}} = a_l^{\mathbb{N}}(L) = |\varphi_l(L)|^2$ where φ_l is a normalized eigenvector associated to λ_l.*

Then, an energy E is a resonance of $H_L^{\mathbb{N}}$ iff

$$S_L(E) := \sum_{l=0}^{L} \frac{a_l^{\mathbb{N}}}{\lambda_l - E} = -e^{-i\theta(E)}, \qquad E = 2\cos\theta(E), \tag{5.1.1}$$

where the determination of $\theta(E)$ is chosen so that $Im\theta(E) > 0$ and $Re\theta(E) \in (-\pi, 0)$ when $ImE > 0$.

Note that the map $E \mapsto \theta(E)$ can be continued analytically from $\mathbb{C}^+$ to the cut plane $\mathbb{C}\backslash((-\infty, 2] \cup [2, +\infty))$ and its continuation is a bijection from $\mathbb{C}\backslash((-\infty, 2] \cup [2, +\infty))$ to $(-\pi, 0) + i\mathbb{R}$. In particular, $\theta(E) \in (-\pi, 0)$ for all $E \in (-2, 2)$.
In Theorem 5.1.1, by taking imaginary parts of two sides of the resonance equation, we obtain that

$$ImS_L(E) = ImE \sum_{j=0}^{L} \frac{a_l^{\mathbb{N}}}{|\lambda_l - E|^2} = e^{ImE} \sin(Re\theta(E)). \tag{5.1.2}$$

Note that, according to the choice of the determination $\theta(E)$ in Theorem 5.1.1, whenever $ImE > 0$, $\sin(Re\theta(E))$ is negative and $ImS_L(E) > 0$. Hence, all resonances of $H_L^{\mathbb{N}}$ lie completely in the lower half-plane $\{ImE < 0\}$.
According to the equation (5.1.1), resonances of $H_L^{\mathbb{N}}$ depend only on the spectral data of the operator H_L i.e., the eigenvalues and corresponding normalized eigenvectors of H_L. In order to solve the equation (5.1.1), it is essential to understand how eigenvalues of H_L (qualitatively or quantitatively) behave and what the magnitudes of $a_l := |\varphi_l(L)|^2$ are in the limit $L \to +\infty$.
Before stating the properties of spectral data of H_L, one defines the *quasi-momentum* of $H^{\mathbb{Z}}$:
Let V be a periodic potential of period p and L be large. For $0 \le k \le p-1$, one defines $\widetilde{T}_k = \widetilde{T}_k(E)$ to be a monodromy matrix for the periodic finite difference operators $H^{\mathbb{Z}}$, that is,

$$\widetilde{T}_k(E) = T_{k+p-1,k}(E) = T_{k+p-1}(E) \dots T_k(E) = \begin{pmatrix} a_p^k(E) & a_p^k(E) \\ a_{p-1}^k(E) & a_{p-1}^k(E) \end{pmatrix} \tag{5.1.3}$$

where $\{T_l(E)\}$ are transfer matrices of $H^{\mathbb{Z}}$:

$$T_l(E) = \begin{pmatrix} E - V_l & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.1.4}$$

Besides, for $k \in \{0, \dots, p-1\}$ we write

$$T_{k-1}(E \dots T_0(E) = \begin{pmatrix} a_k(E) & b_k(E) \\ a_{k-1}(E) & b_{k-1}(E) \end{pmatrix}. \tag{5.1.5}$$

We observe that the coefficients of $\widetilde{T}_k(E)$ are monic polynomials in E. Moreover, $a_p^k(E)$ has degree p and $b_p^k(E)$ has a degree $p-1$. The determinant of $T_l(E)$ equals to 1 for any l, hence, $\det \widetilde{T}_k(E) = 1$. Besides, $k \mapsto \widetilde{T}_k(E)$ is $p-$periodic since V is a $p-$periodic potential. Moreover, for $j < k$

$$\widetilde{T}_k(E) = T_{k,j}(E)\widetilde{T}_j(E)T_{k,j}^{-1}(E).$$

Thus the discriminant $\Delta(E) := \mathrm{tr}\widetilde{T}_k(E) = a_p^k(E) + b_{p-1}^k(E)$ is independent of k and so are $\rho(E)$ and $\rho(E)^{-1}$, eigenvalues of $\widetilde{T}_k(E)$. Now, one can define the Floquet *quasi-momentum* $E \mapsto \theta_p(E)$ by

$$\Delta(E) = \rho(E) + \rho^{-1}(E) = 2\cos\left(p\theta_p(E)\right). \tag{5.1.6}$$

Then, one can show that the spectrum of $H^{\mathbb{Z}}$, $\Sigma_{\mathbb{Z}}$, is the set $\{E||\Delta(E)| \le 2\}$ and

$$\partial\Sigma_{\mathbb{Z}} = \{E||\Delta(E)| = 2 \text{ and } \widetilde{T}_0(E) \text{ is not diagonal}\}.$$

Note that each point of $\partial\Sigma_{\mathbb{Z}}$ is a branch point of $\theta_p(E)$ of square-root type.

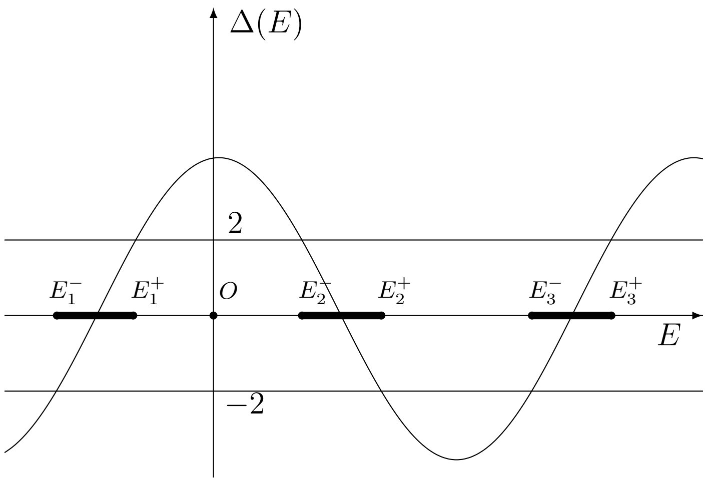


Figure 5.1: Function $\Delta(E)$.

One decomposes $\Sigma_{\mathbb{Z}}$ into its connected components i.e.$\Sigma_{\mathbb{Z}} = \bigcup_{i=1}^q B_i$ with $q < p$. Let c_i be the number of closed gaps contained in B_i. Then, θ_p maps B_i bijectively into $\sum_{\ell=1}^{i-1}(1+c_\ell)\frac{\pi}{p} + \frac{\pi}{p}[0, c_i]$. Moreover, on this set, the derivative of θ_p is proportional to the common density of states $n(E)$ of $H^{\mathbb{Z}}$ and $H^{\mathbb{N}}$:

$$\theta_p'(E) = \pi n(E).$$

One has the following description for spectral data of H_L.

Theorem 5.1.2. *[Klo, Theorem 4.2] For any $j \in \{0, \dots, p-1\}$, there exists $h_j : \Sigma_{\mathbb{Z}} \to \mathbb{R}$, a continuous function that is real analytic in a neighborhood of $\mathring{\Sigma}_{\mathbb{Z}}$ such that, for $L = Np+j$,*

1. *The function h_j maps B_i into $(-(c_i+1)\pi, (c_i+1)\pi)$ where c_i is the number of closed gaps in B_i;*
2. *the function $\theta_{p,L} = \theta_p - \frac{h_j}{L-j}$ is strictly monotonous on each band B_i of $\Sigma_{\mathbb{Z}}$;*
3. *for $1 \le i \le q$, the eigenvalues of H_L in B_i, the ith band of $\Sigma_{\mathbb{Z}}$, says $(\lambda_k^i)_k$ are the solutions (in $\Sigma_{\mathbb{Z}}$) to the quantization condition*

$$\theta_{p,L}(\lambda_k^i) = \frac{k\pi}{L-j}, \quad k \in \mathbb{Z}. \tag{5.1.7}$$

4. *If λ is an eigenvalue of H_L outside $\Sigma_{\mathbb{Z}}$ for $L = Np + j$ large, there exists $\lambda_\infty \in \Sigma_0^+ \cup \Sigma_j^- \backslash \Sigma_{\mathbb{Z}}$ s.t. $|\lambda - \lambda_\infty| \le e^{-cL}$ with $c > 0$ independent of L and λ.*

When solving the equation (5.1.7), one has to do it for each band B_i, and for each band and each k such that $\frac{k\pi}{L-j} \in \theta_{p,L}(B_i)$, (5.1.7) admits a unique solution. But, it may happens that one has two solutions to (5.1.7) for a given k belonging to neighboring bands. Following is the description of the associated eigenfunctions.

Theorem 5.1.3. *[Klo, Theorem 4.3] Recall that $(\lambda_l)_l$ are the eigenvalues of H_L in $\Sigma_{\mathbb{Z}}$ (enumerated as in Theorem 5.1.2).*

1. *There are $p+1$ positive functions, say, f_0^+ and $(f_j^-)_{0\le j\le p-1}$, that are real analytic in a neighborhood of $\mathring{\Sigma}_{\mathbb{Z}}$ such that, for $E_0 \in \mathring{\Sigma}_{\mathbb{Z}}$ such that $\theta_p(E_0) \in [\pi j/p, \pi(j+1)/p)$, there exists $\mathcal{V}$, a complex neighborhood of E_0, real constants m_p, L, $\kappa_{p,L}^0$ and $\kappa_{p,L}^j$ and a function $\Xi_{p,L}$ real analytic $\mathcal{V}$ such that, for $L = Np + k$ sufficiently large, for $\lambda_k \in \mathcal{V}$, one has*

$$\begin{aligned} |\varphi_l(0)|^2 = \frac{f_0(\lambda_l)}{L-k} F_{p,L,j_0}(\lambda_l), \qquad |\varphi_l(L)|^2 = \frac{f_j(\lambda_l)}{f_0(\lambda_l)} |\varphi_l(0)|^2, \\ \varphi_l(L)\overline{\varphi(0)} = \pm e^{i(L-j)\theta_{p,L}(\lambda_l)} \frac{\sqrt{f_0(\lambda_l) f_j(\lambda_l)}}{L-j} F_{p,L,j_0}(\lambda_j), \\ F_{p,L,j_0}(\lambda) = 1 + \kappa g_{m_{p,L}-\pi j}((L-j)\theta_{p,L}(\lambda_l)) + \frac{\Xi_{p,L}(\lambda)}{L-j}, \end{aligned} \tag{5.1.8}$$

where

- *the constant $\kappa_{p,L}$ vanishes except if E_0 is a bad closed gap;*
- *for $m \neq \pi\mathbb{Z}$, the entire function g_m is given by*

$$g_m(z) = \frac{\sin^2(\pi z + m)}{(\pi z + m)\sin^2 m};$$

- *for some* $m_j(E_0) \in (-\pi, 0)$ *and* $\kappa_j(E_0) \in \mathbb{R}$

$$m_{p,L} = m_j(E_0) + O\left(\frac{1}{L-j}\right) \text{ and } \kappa_{p,L} = \kappa_j(E_0) + O\left(\frac{1}{L-j}\right);$$

- *the sign* $\pm$ *is constant on every band of the spectrum* $\Sigma_{\mathbb{Z}}$.

2. *Let* λ *be an eigenvalue outside* $\Sigma_{\mathbb{Z}}$ *and* φ *is a normalized associated eigenvector, one has one of the following alternatives for* $L = Np + j$ *large*
   - *If* $\lambda_\infty \in \Sigma_{\mathbb{N}} \backslash \Sigma_j^-$, *one has*

$$|\varphi(L)| \asymp e^{-cL} \text{ and } |\varphi(0)| \asymp 1;$$

   - *If* $\lambda_\infty \in \Sigma_j^- \backslash \Sigma_{\mathbb{N}}$, *one has*

$$|\varphi(L)| \asymp 1 \text{ and } |\varphi(0)| \asymp e^{-cL};$$

   - *If* $\lambda_\infty \in \Sigma_j^- \cap \Sigma_{\mathbb{N}}$, *one has*

$$|\varphi(L)| \asymp 1 \text{ and } |\varphi(0)| \asymp 1.$$

   Let $\mathcal{B}_j$ *be the set of bad closed gaps for* j *(c.f. [Klo, Proposition 4.1] for more details). Then, there exists* $\tilde{f}$ *real analytic on* $\mathring{\Sigma}_{\mathbb{Z}} \backslash \mathcal{B}_j$ *such that, for eigenvalues, on this set, in (5.1.8), one can take*

$$F_{p,L,j_0}(\lambda) = 1 + \frac{\Xi_{p,L}(\lambda)}{L-j} = \left(1 + \frac{\tilde{f}(\lambda)}{L-j}\right)^{-1}.$$

Remark 5.1.4. *According to [Klo, Section 4], we have the following behavior of* a_k *associated to* λ_k *which is close to* $\partial\Sigma_{\mathbb{Z}}$.
Let $E_0 \in \partial\Sigma_{\mathbb{Z}}$ *and* $L = Np + j$. *We define* $d_{j+1} = a_{j+1}(E_0)(a_p^0(E_0) - \rho^{-1}(E_0)) + b_{j+1}(E_0)a_{p-1}^0(E_0)$ *where* $a_{j+1}, b_{j+1}, a_p^0, a_{p-1}^0$ *are polynomials defined in (5.1.3) and (5.1.5). Then, one distinguishes two cases:*

1. *If* $a_{p-1}^0(E_0) = 0$, *then*

$$a_k = |\varphi_k(L)|^2 \asymp \frac{|\lambda_k - E_0|}{L-j} \text{ and } |\varphi_k(0)|^2 \asymp \frac{1}{L-j}.$$

2. *If* $a_{p-1}^0(E_0) \neq 0$, *then*
   – *if* $d_{j+1} \neq 0$, *one has*

$$|\varphi_k(L)|^2 \asymp \frac{|\lambda_k - E_0|}{L-j} \text{ and } |\varphi_k(0)|^2 \asymp \frac{|\lambda_k - E_0|}{L-j}.$$

   – *if* $d_{j+1} = 0$, *one has*

$$|\varphi_k(L)|^2 \asymp \frac{1}{L-j} \text{ and } |\varphi_k(0)|^2 \asymp \frac{|\lambda_k - E_0|}{L-j}.$$

5.2 The resonance free regions

The first step to describe the asymptotic behavior of resonances as $L \to +\infty$ is to determine a (L-dependent) resonance free region i.e., a region where we can find no resonances for any L large enough. Obviously, one tries to find such a region as large as possible.
We would like to begin the present section with recalling *general estimates for resonance free regions* stated in [Klo]. For the sake of simplicity, one omits the superscript $\mathbb{N}$ from $a_j^{\mathbb{N}}$ in this section when one solves the resonance equation (5.1.1) except that there might be a risk of confusion.

Theorem 5.2.1. *[Klo, Theorem 3.1] Fix $\delta > 0$. Then, there exists $C > 0$ (independent of V and L) such that, for any L and $l \in \{0, \dots, L\}$ such that $-4 + \delta \leq \lambda_{l-1} + \lambda_l < \lambda_{l+1} + \lambda_l \leq 4 - \delta$, the equation (5.1.1) has no solutions in the set*

$$U_l := \left\{ \begin{array}{ll} E \in \mathbb{C}; & ReE \in \left[\frac{\lambda_{l-1}+\lambda_l}{2}, \frac{\lambda_{l+1}+\lambda_l}{2}\right] \\ & 0 \geq C.\theta'_\delta ImE > -a_l d_l^2 |\sin Re\theta(E)| \end{array} \right\} \tag{5.2.1}$$

where the map $E \mapsto \theta(E)$ is defined in Section 5.1 and

$$d_l := \min(\lambda_{l+1} - \lambda_l, \lambda_l - \lambda_{l-1}, 1), \quad \theta'_\delta := \max_{|E| \leq 2 - \frac{\delta}{2}} |\theta'(E)|.$$

Theorem 5.2.1 gives us a quite general estimate on free resonance equation with no restrictions on the quantities d_l and a_l except their being positive. Recall that $a_l = |\varphi_l(L)|^2$ where φ_l is the associated normalized eigenvector of λ_l. We see that, the resonance free region U_l in 5.2.1 depends not much on the spectral data of H_L. The cost of such a generality is, normally, that such a resonance free region is not at all optimal.
Following is another general estimate under an additional assumption on spectral data of H_L: $a_l \ll d_l^2$.

Theorem 5.2.2. *[Klo, Theorem 3.2] Fix $\delta > 0$. Pick $l \in \{0, \dots, L\}$ such that $-4 + \delta \leq \lambda_{l-1} + \lambda_l < \lambda_{l+1} + \lambda_l \leq 4 - \delta$. Then, there exists $C > 0$ (depending only on) such that, for any L, if $a_l \leq d_l^2/C^2$, the equation (5.1.1) has no solutions in the set*

$$\begin{aligned} \widetilde{U}_l := & \left\{ \begin{array}{l} E \in \mathbb{C}; \quad ReE \in \left[\frac{\lambda_{l-1}+\lambda_l}{2}, \lambda_l - Ca_l\right] \cup \left[\lambda_l + Ca_l, \frac{\lambda_{l+1}+\lambda_l}{2}\right] \\ \qquad -Ca_l \leq ImE \leq -a_l d_l^2/C \end{array} \right\} \\ & \bigcup \left\{ \begin{array}{l} E \in \mathbb{C}; \quad ReE \in \left[\frac{\lambda_{l-1}+\lambda_l}{2}, \frac{\lambda_{l+1}+\lambda_l}{2}\right] \\ \qquad -d_l^2/C \leq ImE \leq -Ca_l \end{array} \right\} \end{aligned} \tag{5.2.2}$$

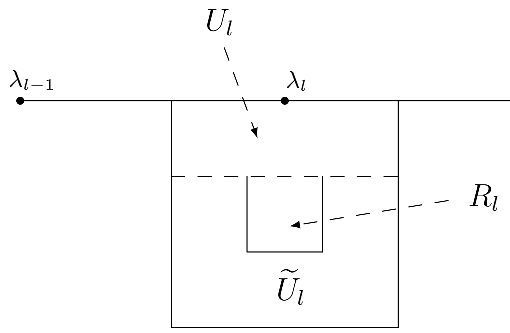


Figure 5.2: The resonance free zones U_l and $\widetilde{U}_l$.

Moreover, under the assumption $a_l \ll d_l^2$, one can resolve the resonance equation (5.1.1) near an eigenvalue λ_l of H_L. Precisely, one proves that, in the rectangle R_l, there exists one and only one resonance and its asymptotic is given in the following theorem:

Theorem 5.2.3. *[Klo, Theorem 3.3] Pick $l \in \{0, \dots, L\}$ such that $-4 < \lambda_{l-1} + \lambda_l < \lambda_{l+1} + \lambda_l < 4$. There exists $C > 1$ (depending on $(\lambda_{l-1} + \lambda_l)$ and $4 - (\lambda_{l+1} + \lambda_l)$) such that, for any L, if $a_l \ll d_l^2/C$, the equation (5.1.1) has exactly one solution in the set*

$$R_l := \left\{ \begin{matrix} E \in \mathbb{C}; & ReE \in \lambda_l + Ca_l[-1, 1] \\ & -Ca_l \leq ImE \leq -a_l d_l^2/C \end{matrix} \right\}. \tag{5.2.3}$$

Moreover, the solution of (5.1.1), say $z_l^{\mathbb{N}}$, satisfies

$$z_l^{\mathbb{N}} = \lambda_l + \frac{a_l^{\mathbb{N}}}{S_{L,l}(\lambda_l) + e^{-i\theta(\lambda_l)}} + O\left((a_l^{\mathbb{N}} d_l^{-1})^2\right) \tag{5.2.4}$$

where $S_{L,l}(\lambda_l) := \sum\limits_{k \neq l} \frac{a_k^{\mathbb{N}}}{\lambda_k - E}$.

According to the above theorem, one obtains the following asymptotic of the width of the resonance $z_l^{\mathbb{N}}$:

$$\mathrm{Im} z_l^{\mathbb{N}} := \frac{a_l^{\mathbb{N}} \sin(\theta(\lambda_l))}{[S_{L,l}(\lambda_l) + \cos(\theta(\lambda_l))]^2 + \sin^2(\theta(\lambda_l))}(1 + o(1)). \tag{5.2.5}$$

Note that, since $\lambda_l \in (-2, 2)$, one has $\theta(\lambda_l) \in (-\pi, 0)$, hence, $\sin(\theta(\lambda_l)) < 0$.

Theorem 5.2.4. *[Klo, Theorem 1.2] Let I be a compact interval in $(-2, 2)$. Then*

1. *If* $I \subset \mathbb{R}\backslash\Sigma_{\mathbb{N}}$*, there exists a constant* C *depending on* I *such that* $H_L^{\mathbb{N}}$ *has no resonances in the rectangle* $\{ReE \in I, ImE \geq -1/C\}$ *for all* L *large;*
2. *If* $I \in \mathring{\Sigma}_{\mathbb{Z}}$*, then, there exists a constant* C *such that* $H_L^{\mathbb{N}}$ *has no resonances in the rectangle* $\{ReE \in I, ImE \geq -1/(CL)\}$*;*
3. *Fix* $0 \leq j \leq p-1$ *and assume that* $\mathring{I} \cap \Sigma_{\mathbb{N}} = I \cap \Sigma_{\mathbb{N}} = \{v_l\}$ *for some isolated eigenvalue* v_l *of* $H^{\mathbb{N}}$ *and* $I \cap \Sigma_{\mathbb{Z}} = \emptyset$*:*
   a) If $I \cap \Sigma_j^- = \emptyset$*, the operator* $H_L^{\mathbb{N}}$ *with* $L = Np + j$ *large has a unique resonance in the rectangle* $\{ReE \in I, -c \leq ImE \leq 0\}$ *for some constant* $c > 0$*; moreover, this resonance, say* z_l *is simple and satisfies* $Imz_l \asymp -e^{-\rho_l L}$ *and* $|z_l - \lambda_l| \asymp e^{-\rho_l L}$ *for some* $\rho_l > 0$ *independent of* L*;*
   b) If $I \cap \Sigma_j^- \neq \emptyset$*,* $H_L^{\mathbb{N}}$ *with* $L = Np + j$ *large has no resonances in the rectangle* $\{ReE \in I, -c \leq ImE \leq 0\}$*.*

We see in the above theorem how the resonance free regions are affected by the nature of the spectrum of $H_L^{\mathbb{N}}$. Below a compact interval of $\mathring{\Sigma}_{\mathbb{Z}} \cap (-2,2)$, the width of resonance free region is in magnitude of $\frac{1}{L}$, hence, so close to the real axis as L large. On the contrary, below a compact interval outside the spectrum $\Sigma_{\mathbb{N}}$, we have to go down in the lower half-plane much more deeply to track down very first resonances of $H_L^{\mathbb{N}}$.
Besides, according to the point (3) in Theorem 5.2.4, each discrete eigenvalue v_j of $H^{\mathbb{N}}$ that is not an eigenvalue of H_j^- generates a resonance for $H_L^{\mathbb{N}}$ exponentially close to v_j, hence, exponentially close to the real axis. By contrast, when v_j is a common eigenvalue of $H^{\mathbb{N}}$ and H_j^-, it may generate a resonance but such a resonance is much farther away from the real axis (at a constant distance independent of L to the real axis).

5.3 Asymptotic of resonances

5.3.1 Location of resonances far from the real axis

According to [Klo, Theorem 5.1], in order to locate resonances under the line $\{\text{Im} z = -\varepsilon\}$ for any $\varepsilon > 0$ fixed, it suffices to study the zeros of the function

$$\Xi_k^-(E) := \int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E} + e^{-i\theta(E)} \tag{5.3.1}$$

$$= \int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E} + \frac{E}{2} + \sqrt{\left(\frac{E}{2}\right)^2 - 1}$$

where, in the second formula, the branch of the square root $z \mapsto \sqrt{z^2-1}$ has positive imaginary part for $z \in (-1,1)$.
The function Ξ_k^- is analytic in $\{\mathrm{Im}E < 0\}$ and in a neighborhood of $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$. Moreover, one can prove that if the potential V is not identical to zero, the function Ξ_k^- has finitely many zeros of finite multiplicity in $\{\mathrm{Im}E < 0\}$ and in $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$ (c.f. [Klo, Proposition 1.2]).
Hence, for any compact interval I in the $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$ and for any $\eta > 0$ sufficiently small, one has:
In the strip $I + i(-\infty, -\eta]$, the resonances of $H_L^{\mathbb{N}}$ lie in $\cup D(E_j, e^{-\eta L})$ where $\{E_j\}$ are zeros of Ξ_k^-. Besides, the number of resonances (counted with multiplicity) in $D(E_j, e^{-\eta L})$ is equal to the order of E_j as a zero of Ξ_k^- (c.f. [Klo, Theorem 1.4]).
Therefore, the total number of resonances below $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$ that do not tend to the real axis as $L \to +\infty$ is finite. The only thing left to study now is the resonances closest to the real axis.

5.3.2 Resonances closest to the real axis

In the present subsection, we make a quick summary of results in [Klo] on resonances closest to the real axis. Let I be a compact interval in $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$. Then, roughly speaking, each eigenvalue $\lambda_l \in I$ of H_L, except for some "special ones", generates a resonance E where $|\mathrm{Im}E| \asymp \frac{1}{L}$.
Let's us lay out this result in more detail. First of all, one introduces the following function which will be useful for the description of the resonances: for $j \in \{0,1,\dots,p-1\}$, one sets

$$c^{\mathbb{N}}(E) := i + \frac{\Xi_k^-(E)}{\pi n_j^-(E)} = \frac{1}{\pi n_j^-(E)}\left(S_j^-(E) + e^{-i\theta(E)}\right).$$

where n_j^- is the density of states of H_j^-.
It was shown in [Klo] that, for any compact interval I in $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$, there exists a neighborhood of I such that, the function $c^{\mathbb{N}}$ is analytic and has a positive imaginary part. Moreover, the function $c^{\mathbb{N}}$ takes the value i only at the zeros of Ξ_k^- (c.f. [Klo, Proposition 1.3]). We will see next that the zeros of $c^{\mathbb{N}} - i$ play an important role in the description of resonances closest to the real axis.
Next, pick a compact interval $I \subset (-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$. Recall that H_L is the operator $H_L^{\mathbb{N}}$ restricted to $[0,L]$ with Dirichlet boundary condition at L and $(\lambda_l)_{0\le l\le L}$ are the eigenvalues of H_L listed in the increasing order.

For $\lambda \in I$, one defines the complex number

$$\tilde{z}_l^{\mathbb{N}} = \lambda_l + \frac{1}{\pi n(\lambda_l) L} \cot^{-1} \circ c^{\mathbb{N}} \left[\lambda_l + \frac{1}{\pi n(\lambda_l) L} \cot^{-1} \circ c^{\mathbb{N}} \left(\lambda_l - i \frac{\log L}{L} \right) \right]$$

where the determination of $\cot^{-1}$ is the inverse of the determination $z \mapsto \cot(z)$ mapping $[0, \pi) \times (-\infty, 0)$ onto $\mathbb{C}^+ \backslash \{i\}$.
Note that

$$-\ln L \lesssim L \cdot \mathrm{Im} \tilde{z}_l^{\mathbb{N}} \lesssim -1 \quad \text{and } 1 \lesssim L \cdot \mathrm{Re}(\tilde{z}_{l+1}^{\mathbb{N}} - \tilde{z}_l^{\mathbb{N}})$$

where the constants are uniform for l such that $\lambda_l \in I$. Then, one obtains the following asymptotic expansion in powers of L^{-1} for resonances $z_l^{\mathbb{N}}$:

Theorem 5.3.1. *[Klo, Proposition 1.4] Let $I \subset (-2, 2) \cap \mathring{\Sigma}_{\mathbb{Z}}$ be a compact interval not containing any zero of $E \mapsto c^{\mathbb{N}}(E) - i$. Then, one has*

$$z_l^{\mathbb{N}} = \lambda_l + \frac{1}{\pi n(\lambda_l) L} \cot^{-1} \circ c^{\mathbb{N}}(\lambda_l) + O\left(\frac{1}{L^2}\right)$$

where the remainder term is uniform in l.

Finally, near the zeros of $c^{\mathbb{N}} - i$, the resonances plunge more deeply into the the lower-half plane and their imaginary parts become of order at most $\frac{\ln L}{L}$:

Theorem 5.3.2. *[Klo, Proposition 1.5] Let $E_0 \in (-2, 2) \cap \mathring{\Sigma}_{\mathbb{Z}}$ be the zero of $E \mapsto c^{\mathbb{N}}(E) - i$ of order q.*
Then, for $\alpha > 0$, for L sufficiently large, if l is such that $|\lambda_l - E_0| \leq L^{-\alpha}$, the resonance $z_l^{\mathbb{N}}$ satisfies

$$Im z_l^{\mathbb{N}} = \frac{q}{2\pi n(\lambda_l)} \cdot \frac{\ln \left(|\lambda_l - E_0|^2 + \left(\frac{q \ln L}{2\pi n(\lambda_l) L} \right)^2 \right)}{2L} \cdot (1 + o(1))$$

where the remainder term is uniform in l such that $|\lambda_l - E_0| \leq L^{-\alpha}$.

CHAPTER 6

RESONANCES OF $H_L^{\mathbb{N}}$ NEAR $\partial\Sigma_{\mathbb{Z}}$ IN THE GENERIC CASE

Contents

Pick $E_0 \in \partial\Sigma_{\mathbb{Z}} \cap (-2,2)$ and $L = Np + j$ where p is the period of the potential V. To fix ideas, let's assume that E_0 is the left endpoint of a band B_i of $\Sigma_{\mathbb{Z}}$.
Recall that $H_L^{\mathbb{N}} = -\Delta + V\mathbb{1}_{[0,L]}$ on $l^2(\mathbb{N})$ with Dirichlet boundary conditions (b.c.) at 0 and H_L is the operator $H_L^{\mathbb{N}}$ restricted to $[0,L]$ with Dirichlet b.c. at L. Let $(\lambda_k)_k$ be eigenvalues of H_L and we put, for each k, $a_k = |\varphi_k(L)|^2$ where φ_k is a normalized eigenvector associated to λ_k.
According to Remark 5.1.4, the parameter a_k associated to $\lambda_k \in \mathring{\Sigma}_{\mathbb{Z}}$ near E_0 can have very different behaviors depending on the potential V: Either $a_k \asymp \frac{|\lambda_k - E_0|}{L}$ (the generic case) or $a_k \asymp \frac{1}{L}$ (the non-generic one). *In the present chapter, we study resonances of* $H_L^{\mathbb{N}}$ *near* $\partial\Sigma_{\mathbb{Z}}$ *in the generic case.*
For $L = Np + j$ large, we define $a_{p-1}^0(E), a_{j+1}(E)$ and d_{j+1} as in (5.1.3), (5.1.5) and Remark 5.1.4. We assume the following generic condition throughout this chapter, for $L = Np + j$,

$$\text{either } \left(a_{p-1}^0(E_0) \neq 0 \text{ and } d_{j+1} \neq 0\right) \text{ or } \left(a_{p-1}^0(E_0) = 0 \text{ and } a_{j+1}(E_0) \neq 0\right). \tag{G}$$

Note that, since [Klo, Lemma 4.2], for $L = Np + j$ large, $\partial\Sigma_{\mathbb{Z}} \cap \sigma(H_L) = \{E_0; a_{p-1}^0(E_0) = a_{j+1}(E_0) = 0 \text{ and } b_p^0(E) \neq 0\}$. Therefore, in the generic case, E_0 is not an eigenvalue of H_L when L is large.
Recall that all resonances whose real parts belong to a compact set in $(-2,2) \cap \mathring{\Sigma}_{\mathbb{Z}}$ were well studied in [Klo]. Throughout this chapter, we will study resonances E of the resonance equation (5.1.1) in the rectangle $\mathcal{D} = [E_0, E_0 + \varepsilon_1] - i[0, \varepsilon_2]$ where $\varepsilon_1 \asymp \varepsilon^2$ and $\varepsilon_2 \asymp \varepsilon^5$ with $\varepsilon > 0$ small. To study resonances below compact intervals inside $\Sigma_{\mathbb{Z}}$, the author in [Klo] introduced an analytical method to simplify and resolve the equation (5.1.1) (see [Klo, Theorem 5.1]). Unfortunately, such a method was efficient inside $\Sigma_{\mathbb{Z}}$ but does not seem to work near $\partial\Sigma_{\mathbb{Z}}$. A different approach is thus needed to study resonances near $\partial\Sigma_{\mathbb{Z}}$. We figure out that, near the boundary of $\Sigma_{\mathbb{Z}}$, the spectral data $\{\lambda_k\}_k, \{a_k\}_k$ possess special properties. We exploit them to approximate $S_L(E)$. Concretely, for each eigenvalue λ_k of H_L near $\partial\Sigma_{\mathbb{Z}}$, we approximate $S_L(E)$ in a domain close to λ_k by keeping the term $\frac{a_k}{\lambda_k - E}$ and replacing the sum of the other terms by $\sum\limits_{\ell \neq k} \frac{a_\ell}{\lambda_\ell - \lambda_k}$. Then, we use the Rouché's theorem to describe resonances. For a domain farther from λ_k, we use the (generic) behavior of spectral data again to show that there are no resonances there.
Surprisingly, this method has a flavor of the one used in describing resonances for operators with random potentials and resonances near isolated eigenvalues of $H^{\mathbb{N}}$ with periodic potentials (see Theorem 5.2.3). Nonetheless, in the present case, the situation is different

in many aspects. The hypotheses in Theorem 5.2.3 obviously do not hold in our case and a modification of the proofs in [Klo] could not be a solution either.
In the present chapter, we prove

Theorem 6.0.3. *Assume that $E_0 \in (-2,2)$ is the left endpoint of the ith band B_i of $\Sigma_{\mathbb{Z}}$. We enumerate the spectral data λ_k and a_k in B_i as $(\lambda_\ell^i)_\ell, (a_\ell^i)_\ell$ with $0 \le \ell \le n_i$ (the (local) enumeration w.r.t. bands of $\Sigma_{\mathbb{Z}}$).*
Let $I = [E_0, E_0 + \varepsilon_1]$ and $\mathcal{D} = [E_0, E_0 + \varepsilon_1] - i[0, \varepsilon_2]$ where $\varepsilon_1 \asymp \varepsilon^2$ and $\varepsilon_2 \asymp \varepsilon^5$ with $\varepsilon > 0$ small. Then, we have

1. *For each eigenvalue $\lambda_n^i \in I$ of H_L, there exists a unique resonance z_n in $\mathcal{B}_{n,\varepsilon} = \left[\frac{\lambda_{n-1}^i+\lambda_n^i}{2}, \frac{\lambda_n^i+\lambda_{n+1}^i}{2}\right] - i\left[0, \varepsilon^5\right]$ with a convention that $\lambda_{-1}^i := 2E_0 - \lambda_0$. Moreover, $z_n \in \mathcal{M}_n = \left[\frac{\lambda_{n-1}^i+\lambda_n^i}{2}, \frac{\lambda_n^i+\lambda_{n+1}^i}{2}\right] - i\left[0, C_0\frac{n+1}{L^2}\right]$ with $C_0 > 0$ large. Besides, there are no resonances in the rectangle $[E_0 - \varepsilon, E_0] - i\left[0, C_0\frac{n+1}{L^2}\right]$.*

2. *Define $S_{n,L}^i(E) = S_L(E) - \frac{a_n^i}{\lambda_n^i - E}$ and $\alpha_n = S_{n,L}^i(\lambda_n^i) + e^{-i\theta(\lambda_n^i)}$. Then, there exists $c_0 > 0$ s.t. $c_0 \le |\alpha_n| \lesssim \frac{1}{\varepsilon^2}$ and*

$$z_n = \lambda_n^i + \frac{a_n^i}{\alpha_n} + O\left(\frac{(n+1)^4}{L^5|\alpha_n|^3}\right). \tag{6.0.1}$$

3. *$Im z_n$ satisfies*

$$Im z_n = \frac{a_n^i \sin(\theta(\lambda_n^i))}{|\alpha_n|^2} + O\left(\frac{(n+1)^4}{L^5|\alpha_n|^3}\right). \tag{6.0.2}$$

   Consequently, there exists a large constant $C > 0$ such that $\frac{\varepsilon^4(n+1)^2}{CL^3} \le |Im z_n| \le C\frac{(n+1)^2}{L^3}$.

The above theorem calls for a few comments. First of all, we see that, near $\partial\Sigma_{\mathbb{Z}}$, each eigenvalue $\lambda_n^i \in \overset{\circ}{\Sigma}_{\mathbb{Z}}$ generates a unique resonance z_n which can be described by the formula (6.0.1). Moreover, $|\text{Im} z_n| \asymp \frac{n^2}{L^3}$. Hence, when $n \asymp \varepsilon L$ (corresponding to resonances far from the boundary), $|\text{Im} z_n|$ is of order $\frac{1}{L}$. However, when n is small i.e. λ_n^i is close to the boundary of the spectrum, $|\text{Im} z_n|$ becomes much smaller. Precisely, the magnitude of the width of resonances varies from $\frac{1}{L^3}$ to $\frac{1}{L}$.

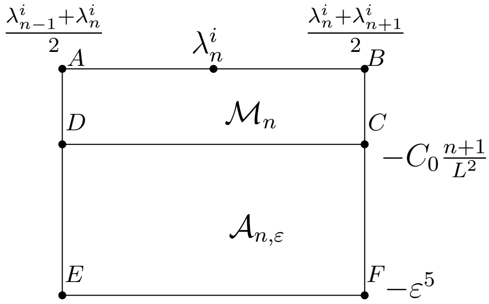


Figure 6.1: Rectangle $\mathcal{B}_{n,\varepsilon}$

Next, we turn our attention to resonances below $\mathbb{R}\backslash\Sigma_{\mathbb{N}}$. Recall that $\Sigma_{\mathbb{N}}$, the spectrum of $H^{\mathbb{N}}$, is the union of $\Sigma_{\mathbb{Z}}$ and isolated simple eigenvalues $\{v_\ell\}_\ell$ of the operator $H^{\mathbb{N}}$.
Let I be a compact interval in $(-2,2)$ and $I\subset\mathbb{R}\backslash\Sigma_{\mathbb{N}}$. Then, according to Theorem 5.2.4, H_L has no resonances in the rectangle $I-i[0,c]$ for some constant $c>0$. In other words, there exists a resonance free region of width at least of order 1 below the compact interval I. This result is a direct consequence of the fact that $\text{dist}(E,\sigma(H_L))$ and $|\text{Im}e^{-i\theta(E)}|$ are lower bounded by a positive constant for $E\in I\subset\mathbb{R}\backslash\Sigma_{\mathbb{N}}\cap(-2,2)$.
In the present chapter, we extend the above result for compact intervals I which meet the boundary of $\Sigma_{\mathbb{Z}}$.

Theorem 6.0.4. *Let $E_0\in(-2,2)$ be the left endpoint of the i-th band B_i of $\Sigma_{\mathbb{Z}}$. Pick $L\in\mathbb{N}^*$ large. Then, $H_L^{\mathbb{N}}$ has no resonances in the rectangle $[E_0-\varepsilon,E_0]-i[0,\varepsilon^5]$ if ε is sufficiently small.*

The structure of the present chapter is as follows. For each $\lambda^i_n\in I=[E_0,E_0+\varepsilon_1]$ with $\varepsilon_1\asymp\varepsilon^2$, we study the resonance equation (5.1.1) in the rectangle $\mathcal{B}_{n,\varepsilon}=\left[\frac{\lambda^i_{n-1}+\lambda^i_n}{2},\frac{\lambda^i_n+\lambda^i_{n+1}}{2}\right]-i\left[0,\varepsilon^5\right]$ (see Figure 6.1). First of all, we study the behavior of spectral data $\{\lambda_k\},\{a_k\}$ near the boundary point E_0 in Section 6.1. Next, in Section 6.2, we show that, in $\mathcal{A}_{n,\varepsilon}$ where $C_0\frac{n+1}{L^2}\leq|\text{Im}E|\leq\varepsilon^5$ with $C_0>0$ sufficiently large, $|\text{Im}S_L(E)|$ will be too small. As a result, there are no resonances in $\mathcal{A}_{n,\varepsilon}$. After that, we will state the proof of Theorem 6.0.3 in Section 6.3. Finally, in the last section, Section 6.4, a proof for Theorem 6.0.4 is given.
Notations: Throughout the present chapter, we will write C for constants whose values can vary from line to line. Constants marked C_i are fixed within a given argument. We

write $a \lesssim b$ if there exists some $C > 0$ independent of parameters coming into a, b s.t. $a \leq Cb$. Finally, $a \asymp b$ means $a \lesssim b$ and $b \lesssim a$.

6.1 Generic behavior of spectral data near the boundary of the spectrum

Let B_i be a band of $\Sigma_{\mathbb{Z}}$ and $(\lambda_n^i)_n$ be (distinct) eigenvalues of H_L in B_i. Pick E_0 an endpoint of B_i and $L = Np + j$.
Then, according to Theorems 5.1.2 and 5.1.3, the spectral data i.e., the eigenvalues $\lambda_\ell^i \in B_i$ and the associated a_ℓ^i, close to E_0, can be represented as $\lambda_\ell^i = \theta_{p,L}^{-1}\left(\frac{\pi \ell}{L-j}\right)$ if $\frac{\pi \ell}{L-j} \in \theta_{p,L}(B_i)$ and $a_\ell^i = \frac{1}{L-j} q(\lambda_\ell^i)$ for some function q. In the present section, we will study the smoothness of the two functions $\theta_{p,L}^{-1}$ and q near $\theta_{p,L}(E_0)$ and E_0 respectively. We will make use of the results in this section to prove a good upper bound for the sum $S_{n,L}^i(\lambda_n^i)$ defined in Theorem 6.0.3 for each $\lambda_n^i \in B_i$. Such an estimate plays an important role in describing the exact magnitude of the imaginary part of resonances near the boundary of $\Sigma_{\mathbb{Z}}$. Finally, we will study the behavior of eigenvalues inside B_i and close to the boundary point E_0.
Recall that, for $L = Np + j$, $\theta_{p,L}(E) = \theta_p(E) - \frac{h_j(E)}{L-j}$ where $\theta_p(E)$ is the quasi-momentum of $H^{\mathbb{Z}}$ and $h_j(E)$ is analytic in $\mathring{\Sigma}_{\mathbb{Z}}$ and satisfies the following relation

$$e^{2ih_j(E)} = \frac{a_{j+1}(E)\left(\rho(E) - a_p^0(E)\right) - b_{j+1}(E) a_{p-1}^0(E)}{\overline{a_{j+1}(E)\left(\rho(E) - a_p^0(E)\right) - b_{j+1}(E) a_{p-1}^0(E)}}.$$

For $0 \leq m \leq p-1$, we define $h_{m-1}(E)$ in the same way

$$e^{2ih_{m-1}(E)} = \frac{a_m(E)\left(\rho(E) - a_p^0(E)\right) - b_m(E) a_{p-1}^0(E)}{\overline{a_m(E)\left(\rho(E) - a_p^0(E)\right) - b_m(E) a_{p-1}^0(E)}}.$$

Here, $\rho(E) = e^{ip\theta_p(E)}$ and $a_{p-1}^0, a_p^0, a_{j+1}, b_{j+1}, a_m, b_m$ are polynomials defined in (5.1.3), (5.1.5), Chapter 5.
First of all, we prove the smoothness of $\theta_{p,L}^{-1}$.

Lemma 6.1.1. *Let $E_0 \in \partial\Sigma_{\mathbb{Z}}$ and B_i be the band of $\Sigma_{\mathbb{Z}}$ containing E_0. We put $J = \theta_{p,L}(B_i)$. Then, $\theta_{p,L}^{-1}$ is C^2 on J and its two first derivatives on J are bounded by a constant independent of L. Besides, $\frac{d}{dx}\theta_{p,L}^{-1}(x) = 0$ and $\frac{d^2}{dx^2}\theta_{p,L}^{-1}(x) \neq 0$ at $x = \theta_{p,L}(E_0)$.*

Proof of Lemma 6.1.1. Assume that $L = Np + j$ where p is the period of the potential V and $0 \leq j \leq p-1$. Since Theorem 5.1.2, $\theta_{p,L}$ is continuous and strictly monotonous on

B_i. Hence, J is a compact interval.
We can assume that E_0 is the left endpoint of the band B_i. Pick $x \in J$. Let $u \in B_i$ such that $\theta_{p,L}(u) = x$. Then, $\frac{d}{dx}\theta_{p,L}^{-1}(x) = \frac{1}{\theta'_{p,L}(u)}$. Note that $\theta_{p,L}(u)$ is analytic and strictly positive in the interior of the band B_i (c.f. Theorem 5.1.2). Hence, it suffices to prove the lemma for u near E_0.
It is well known that, for u near E_0, $\theta'_p(u) = \pi n(u) = c_1|u - E_0|^{-1/2}(1 + o(1))$ and $n'(u) = c_2|u - E_0|^{-3/2}(1 + o(1))$ where $n(u)$ is the density of state of the operator $H^{\mathbb{Z}}$ and $c_1, c_2 \neq 0$ (c.f. [Tes00]).
Put $u = E_0 + t^2$ with $t > 0$. From the definition of h_j, we see that $t \mapsto h_j(E_0 + t^2)$ is analytic near 0. Indeed, we put $s(u) = a_{j+1}(u)\left(\rho(u) - a_p^0(u)\right) - b_{j+1}(u)a_{p-1}^0(u)$ where $\rho(u) = e^{ip\theta_p(u)}$. Then, since $\rho(E_0 + t^2)$ is analytic near 0, so is $s(E_0 + t^2)$. We rewrite $s(E_0 + t^2) = \alpha(t) + i\beta(t)$ where $\alpha(t) = c_\alpha t^{k_1}(1 + g_1(t))$ and $\beta(t) = c_\beta t^{k_2}(1 + g_2(t))$ with $c_\alpha, c_\beta \neq 0$, $k_1, k_2 \in \mathbb{N}$ and $g_1(t), g_2(t)$ analytic near 0. W.o.l.g., assume that $c_\alpha = c_\beta = 1$ and $k_1 \geq k_2$. Hence,

$$e^{2ih_j(E_0+t^2)} = \frac{t^{2(k_1-k_2)}(1 + g_1(t))^2 - (1 + g_2(t))^2 + 2it^{2k_1}(1 + g_1(t))(1 + g_2(t))}{t^{2(k_1-k_2)}(1 + g_1(t))^2 + (1 + g_2(t))^2}.$$

This formula implies directly that $h_j(E_0 + t^2)$ is analytic in t near 0.
Consequently, $h'_j(u) = O\left(|u - E_0|^{-1/2}\right)$ and $h''_j(u) = O\left(|E - E_0|^{-3/2}\right)$.
Therefore, $\frac{d}{dx}\theta_{p,L}^{-1}(x) = c|u - E_0|^{1/2}(1 + o(1))$ near $\theta_{p,L}(E_0)$ where $c \neq 0$. In the other words, $\frac{d}{dx}\theta_{p,L}^{-1}(x)$ is continuous and bounded by a constant independent of L on J. Moreover, $\frac{d}{dx}\theta_{p,L}^{-1}(x) = 0$ at $x = \theta_{p,L}(E_0)$.
Next, we study the second derivarive of $\theta_{p,L}^{-1}(x)$ on the interval J. We compute

$$\frac{d^2}{dx^2}\theta_{p,L}^{-1}(x) = -\frac{\theta''_{p,L}(u)}{\left(\theta'_{p,L}(u)\right)^3} = \frac{-\pi n'(u) + \frac{h''_j(u)}{L-j}}{\left(\pi n(u) - \frac{h'_j(u)}{L-j}\right)^3}. \tag{6.1.1}$$

We observe that the numerator of the right hand side (RHS) of (6.1.1) is equal to $c_3|u - E_0|^{-3/2}(1 + o(1))$ and the denominator of RHS of (6.1.1) is equal to $c_4|u - E_0|^{-3/2}(1 + o(1))$ near E_0 where c_3, c_4 are non-zero. Hence, $\theta_{p,L}^{-1}(x)$ is C^2 on the whole interval J and $\frac{d^2}{dx^2}\theta_{p,L}^{-1}(x) \neq 0$ at $x = \theta_{p,L}(E_0)$. Moreover, its second derivative is bounded on J by a positive constant independent of L. □

The following lemma will be useful for proving the smoothness of a_k as a function of λ_k.

Lemma 6.1.2. *Let E_0 be one endpoint of a band B_i of $\Sigma_{\mathbb{Z}}$. For $0 \le m \le p-1$, we define, on the band B_i,*

$$\xi_m(E) = \cos\left(h_j(E) - p\theta_p(E) - 2h_{m-1}(E)\right) \cdot \frac{\sin\left(h_j(E)\right)}{\sin\left(p\theta_p(E)\right)} \tag{6.1.2}$$

and $\nu_m(E) = 1 - \cos\left(2h_j(E) - 2h_{m-1}(E)\right)$.
Then, ξ_m and ν_m are analytic near 0 as a function of the variable $t = \sqrt{|E - E_0|}$. Moreover, $\nu_m(E) = \nu_{m,0} + \nu_{m,2}t^2 + O(t^3)$ where $\nu_{m,0}$ is equal to either 0 or 2, $\nu_{m,2} \in \mathbb{R}$.

Proof of Lemma 6.1.2. W.o.l.g., we assume that E_0 is the left endpoint of B_i. Then, we write $E = E_0 + t^2$ for $t > 0$. Let's consider the function $\xi_m(E)$ first. Note that $h_j(E_0 + t^2), h_{m-1}(E_0 + t^2)$ are analytic in t near 0 (see the proof of Lemma 6.1.1). Hence, we can write

$$h_j(E_0 + t^2) = h_j(E_0) + h_{j,1}t + h_{j,2}t^2 + O(t^3); \tag{6.1.3}$$

$$h_{m-1}(E_0 + t^2) = h_{m-1}(E_0) + h_{m-1,1}t + h_{m-1,2}t^2 + O(t^3) \tag{6.1.4}$$

where $h_j(E_0), h_{m-1}(E_0)$ belong to either $\pi\mathbb{Z}$ or $\frac{\pi}{2} + \pi\mathbb{Z}$ (see [Klo, Lemma 4.4]).
First case: Assume that $h_j(E_0) \in \pi\mathbb{Z}$.
By Taylor's expansion for the sine function, we have

$$\sin\left(h_j(E)\right) = \pm\left(h_{j,1}t + h_{j,2}t^2\right) + O(t^3). \tag{6.1.5}$$

W.o.l.g., assume that $\sin\left(h_j(E)\right) = h_{j,1}t + h_{j,2}t^2 + O(t^3)$.
Next, we write $\theta_p(E) = \theta_{p,0} + \theta_{p,1}t + O(t^2)$ where $\theta_{p,0} = \theta_p(E_0) \in \frac{\pi}{p}\mathbb{Z}$ and $\theta_{p,1} \neq 0$. Note that $p\theta_p(E_0), 2h_{m-1}(E_0) \in \pi\mathbb{Z}$, hence, $h_j(E_0) - p\theta_p(E_0) - 2h_{m-1}(E_0) \in \pi\mathbb{Z}$. Then, Taylor's expansion for the cosine function yields

$$\cos\left(h_j(E) - p\theta_p(E) - 2h_{m-1}(E)\right) = \pm 1 \mp \frac{t^2}{2}(h_{j,1} - p\theta_{p,1} - 2h_{m-1,1})^2 + O(t^3). \tag{6.1.6}$$

Hence, (6.1.5)-(6.1.6) yield

$$\cos\left(h_j(E) - p\theta_p(E) - 2h_{m-1}(E)\right) \cdot \sin\left(h_j(E)\right) = \epsilon h_{j,1}t + \epsilon h_{j,2}t^2 + O(t^3) \tag{6.1.7}$$

where $\epsilon = \pm 1$.
On the other hand, since $p\theta_p(E_0) \in \pi\mathbb{Z}$, we have

$$\sin\left(p\theta_p(E)\right) = \pm p\theta_{p,1}t + O(t^3). \tag{6.1.8}$$

Thanks to (6.1.7)-(6.1.8), we infer that

$$\xi_m(E) = \cos\left(h_j(E) - p\theta_p(E) - 2h_{m-1}(E)\right) \cdot \frac{\sin\left(h_j(E)\right)}{\sin\left(p\theta_p(E)\right)} = \xi_{m,0} + \xi_{m,1}t + O(t^2) \quad (6.1.9)$$

where $\xi_{m,0}$ and $\xi_{m,1}$ are independent of m.

Note that, all functions which we have considered so far are analytic in t. Hence, $O(t^2)$ in (6.1.9) can be written in $t^2 g(t)$ where $g(t)$ is analytic near 0. Hence, the above asymptotic shows that ξ_m is analytic near 0 as a function of t.

Second case: $h_j(E_0) \in \frac{\pi}{2} + \pi\mathbb{Z}$.

Note that, when we use the Taylor expansions in this case, the roles of the sine and the cosine terms in $\xi_m(E)$ are interchanged. Hence, $\xi_m(E)$ can be written in the same form as in (6.1.9) but with different coefficients $\xi_{m,0}$ and $\xi_{m,1}$. In this case $\xi_{m,0}$ depends on m. Hence, $t \mapsto \xi_m(E_0 + t^2)$ is analytic near 0.

Finally, we consider the function $\nu_m(E)$. Since h_j, h_{m-1} are analytic in t near 0, so is ν_m. On the other hand, $2h_j(E_0), 2h_{m-1}(E_0)$ always belong to $\pi\mathbb{Z}$. Hence, from Taylor's expansion of the function cosine, it is easy to see that the the coefficient of order 1 in Taylor's expansion of $\nu_m(E_0+t^2)$ vanishes. Precisely, $\nu_m(E) = \nu_{m,0} + \nu_{m,2}t^2 + O(t^3)$ where $\nu_{m,0}$ is equal to either 0 or 2. □

Now we prove the smoothness of a_k as a function of λ_k.

Lemma 6.1.3. *Let E_0 be an endpoint of one band B_i of $\Sigma_{\mathbb{Z}}$ and $L = Np + j$. Assume the condition (G). Then, for each eigenvalue λ_k near E_0, $a_k = \frac{1}{L-j}q(\lambda_k)$ where q is a C^1 function near E_0 and q is bounded near E_0 by a constant C_{lip} independent of L.*
Consequently, for all λ_k, λ_n near E_0, we have

$$|a_k - a_n| \le \frac{C_{lip}}{L}|\lambda_k - \lambda_n|. \quad (6.1.10)$$

Proof of Lemma 6.1.3. Pick an eigenvalue $\lambda_k \in \overset{\circ}{\Sigma}_{\mathbb{Z}}$ close to E_0. Then, according to Theorem 5.1.3, we have $a_k = \frac{1}{L-j}q(\lambda_k)$ where $L = Np + j$ and

$$q(E) = \frac{|a^0_{p-1}(E)|^2}{\left|a_{j+1}(E)\left(a^0_p(E) - \rho^{-1}(E)\right) + b_{j+1}(E)a^0_{p-1}(E)\right|^2} f^{-1}(E)\left(1 + \frac{\tilde{f}(E)}{L-j}\right)^{-1}. \quad (6.1.11)$$

To prove the present lemma, it suffices to show that the derivative $q'(E)$ is continuous up to E_0 and $q'(E)$ is bounded near E_0 by a constant independent of L. Note that our

function $q(E)$ depends on L.
We define, for $0 \le m \le p-1$,

$$v_m(E) = a_m(E)\left(a_p^0(E) - \rho^{-1}(E)\right) + b_m(E)a_{p-1}^0(E)$$

and $\psi_m(E) = \mathrm{Re}\,(v_m(E)) = a_m(E)\left(a_p^0(E) - \cos(p\theta_p(E))\right) + b_m(E)a_{p-1}^0(E)$. Since $\theta_p(E) - \theta_p(E_0) = c_1|E-E_0|^{1/2}(1+o(1))$, $\theta_p'(E) = \pi n(E) = c_2|E-E_0|^{-1/2}(1+o(1))$ with $c_1, c_2 \neq 0$ and $\sin(p\theta_p(E_0)) = 0$, we infer that $\psi_m(E)$ is a C^1 function up to E_0. Consequently, $|v_m(E)|^2 = \psi_m^2(E) + a_m^2(E)\sin^2(p\theta_p(E))$ is C^1 on the band B_i of $\Sigma_{\mathbb{Z}}$.
W.o.l.g., assume that E_0 is the left endpoint of the band B_i. We write $E = E_0 + t^2$ for $t > 0$. Then, $\rho(E_0+t^2), \theta_p(E_0+t^2)$ are analytic in t near 0.
Recall that $f(E) = \frac{2}{p}\sum_{m=0}^{p-1}|\alpha_m(E)|^2$ where $\alpha_m(E) = \frac{v_m(E)}{\rho(E)-\rho^{-1}(E)} = -i\frac{v_m(E)}{2\sin(p\theta_p(E))}$ and

$$\begin{aligned}\tilde{f}(E) &= \frac{2}{f(E)}\left[\sum_{m=0}^{p-1}|\alpha_m(E)|^2\cos(h_j(E) - p\theta_p(E) - 2h_{m-1}(E))\right]\cdot\frac{\sin(h_j(E))}{\sin(p\theta_p(E))}\\ &+ \frac{2}{f(E)}\sum_{m=0}^{j}|\alpha_m(E)|^2\left(1-\cos\left(2\left(h_j(E) - h_{m-1}(E)\right)\right)\right)\\ &= \frac{2}{f(E)}\sum_{m=0}^{p-1}|\alpha_m(E)|^2\xi_m(E) + \frac{2}{f(E)}\sum_{m=0}^{j}|\alpha_m(E)|^2\nu_m(E).\end{aligned}$$

(c.f. [Klo, Section 4.1.4]).
We compute

$$|\alpha_m(E)|^2 = \frac{1}{4\sin^2(p\theta_p(E))}\left(a_m^2(E)\sin^2(p\theta_p(E)) + \psi_m^2(E)\right). \tag{6.1.12}$$

Recall that, since (6.1.8), we can represent $\sin(p\theta_p(E)) = \pm p\theta_{p,1}t + O(t^3)$ with $\theta_{p,1} \neq 0$.
Next, since $a_m(E)$ is a polynomial and $\psi_m(E)$ is C^1 up to E_0, we can write

$$a_m(E_0+t^2) = a_m(E_0) + a_m'(E_0)t^2 + O(t^4); \tag{6.1.13}$$

$$\psi_m(E_0+t^2) = \psi_m(E_0) + \psi_{m,2}t^2 + O(t^3). \tag{6.1.14}$$

Plugging (6.1.8), (6.1.13) and (6.1.14) in (6.1.12), we obtain

$$|\alpha_m(E)|^2 = \frac{1}{t^2}\left(\alpha_{m,0} + \alpha_{m,2}t^2 + O(t^3)\right) \tag{6.1.15}$$

where $\alpha_{m,0} = \frac{\psi_m^2(E_0)}{4p^2\theta_{p,1}^2}$, $\alpha_{m,2} = \frac{1}{4p^2\theta_{p,1}^2}\left(p^2\theta_{p,1}^2 a_m^2(E_0) + 2\psi_m(E_0)\psi_{m,2}\right)$. Hence,

$$f(E_0+t^2) = \frac{1}{t^2}\left(f_0 + f_2 t^2 + O(t^3)\right) \tag{6.1.16}$$

where $f_0 = \frac{1}{2p^3\theta_{p,1}^2}\sum_{m=0}^{p-1}\psi_m^2(E_0)$ and

$$f_2 = \frac{1}{2p^3\theta_{p,1}^2}\left(p^2\theta_{p,1}^2\sum_{m=0}^{p-1}a_m^2(E_0) + 2\sum_{m=0}^{p-1}\psi_m(E_0)\psi_{m,2}\right).$$

For $0 \le m \le p-1$, let $\xi_m(E)$ be the function defined in (6.1.2), Lemma 6.1.2. Then, $\xi_m(E_0+t^2)$ is analytic in t near 0 and we write $\xi_m(E) = \xi_{m,0} + \xi_{m,1}t + O(t^2)$. Hence, (6.1.15) yield

$$2\sum_{m=0}^{p-1}|\alpha_m(E)|^2\xi_m(E) = \frac{1}{t^2}\left(\beta_0 + \beta_1 t + O(t^2)\right) \tag{6.1.17}$$

where $\beta_0 = 2\sum_{m=0}^{p-1}\alpha_{m,0}\xi_{m,0}$ and $\beta_1 = 2\sum_{m=0}^{p-1}\alpha_{m,0}\xi_{m,1}$.

Note that all series expansions in t which we have used so far are associated to analytic functions in t. We thus can write $O(t^2)$ in (6.1.17) as $\beta_2 t^2 + t^3 g_0(t)$ for some $\beta_2 \in \mathbb{R}$ and g_0 analytic near 0.

Now we put $\nu_m(E) = 1 - \cos\left(2h_j(E) - 2h_{m-1}(E)\right)$. Thanks to Lemma 6.1.2, we can write $\nu_m(E) = \nu_{m,0} + \nu_{m,2}t^2 + O(t^3)$. Then,

$$2\sum_{m=0}^{j}|\alpha_m(E)|^2\nu_m(E) = \frac{1}{t^2}\left(\gamma_0 + \gamma_2 t^2 + O(t^3)\right) \tag{6.1.18}$$

where $\gamma_0 = 2\sum_{m=0}^{j}\alpha_{m,0}\nu_{m,0}$.

To sum up,

$$f(E)\tilde{f}(E) = \frac{1}{t^2}\left(\beta_0 + \gamma_0 + \beta_1 t + (\beta_2+\gamma_2)t^2 + t^3 g_1(t)\right)$$

with $g_1(t)$ analytic near 0.

Hence, for $L = Np + j$ large, we have

$$f(E) + \frac{1}{L-j}f(E)\tilde{f}(E) = \frac{1}{t^2}\left[\left(f_0 + \frac{\beta_0+\gamma_0}{L-j}\right) + \frac{\beta_1}{L-j}t + \left(f_2 + \frac{\beta_2+\gamma_2}{L-j}\right)t^2 + t^3 g_2(t)\right] \tag{6.1.19}$$

where $g_2(t)$ is analytic near 0. Moreover, $g_2(t)$ and its derivative are bounded by a constant independent of L .

According to the condition (G), we distinguish two cases:

First case: Assume that $a^0_{p-1}(E_0) \neq 0$ and $d_{j+1} = v_{j+1}(E_0) \neq 0$.

First of all, in the present case, the function $\frac{|a^0_{p-1}(E)|^2}{\left|a_{j+1}(E)\left(a^0_p(E)-\rho^{-1}(E)\right)+b_{j+1}(E)a^0_{p-1}(E)\right|^2} = \frac{|a^0_{p-1}(E)|^2}{|v_{j+1}(E)|^2}$ is analytic in t near 0.

Next, from the definition of $a_m(E)$ and $b_m(E)$ (see (5.1.5), Chapter 5), we have

$$\begin{vmatrix} a_m(E) & b_m(E) \\ a_{m-1}(E) & b_{m-1}(E) \end{vmatrix} = 1. \tag{6.1.20}$$

Combining this with the hypothesis that $a^0_{p-1}(E_0) \neq 0$, we infer that there exists $m_0 \in \{0, \ldots, p-1\}$ s.t. $\psi_m(E_0) \neq 0$. Hence, $f_0 > 0$. Then, with $L = Np + j$ sufficiently large, we can represent

$$f^{-1}(E)\left(1 + \frac{\tilde{f}(E)}{L-j}\right)^{-1} = \frac{t^2}{f_{0,L} + tg_3(t)} \tag{6.1.21}$$

where the analytic function $g_3(t)$ and its derivative are bounded by a positive constant independent of L. Moreover, $f_{0,L}$ is lower bounded and upper bounded by positive constants independent of L. Hence, $q(E)$ can be written as $t^2 g_L(t)$ where $g_L(t)$ is analytic near 0. The function $g_L(t)$ does not vanish at 0 and $\max\{|g_L(t), g'_L(t)|\} \leq C$ with some $C > 0$ independent of L.

Second case: Assume that $a^0_{p-1}(E_0) = 0$ and $a_{j+1}(E_0) \neq 0$

Since $a^0_{p-1}(E_0) = 0$, the monodromy matrix $\widetilde{T}_0(E_0)$ defined in (5.1.3) is upper triangular with eigenvalues $\rho(E_0) = \rho^{-1}(E_0) = \pm 1$. Hence, $a^0_p(E_0) - \rho^{-1}(E_0) = 0$. Then, $\psi_m(E_0)$, hence $\alpha_{m,0}$, is equal to 0 for every $0 \leq m \leq p-1$. Consequently, $f_0 = \beta_0 = \gamma_0 = \beta_1 = 0$. On the other hand, in the present case, $f_2 = \frac{1}{2p}\sum_{m=0}^{p-1} a^2_m(E_0)$. Note that, by (6.1.20), $a_m(E)$ and $a_{m-1}(E)$ can not vanish at E_0 simultaneously. Therefore, f_2 is strictly positive and

$$f^{-1}(E)\left(1 + \frac{\tilde{f}(E)}{L-j}\right)^{-1} = \frac{1}{f_{2,L} + tg_4(t)} \tag{6.1.22}$$

where $g_4(t)$ is analytic near 0. Moreover, there exists $C > 0$ independent of L such that $\max\{|g_4(t)|, |g'_4(t)|\} \leq C$ near 0 and $\frac{1}{C} \leq f_{2,L} \leq C$.

Now we study the function $\frac{|a^0_{p-1}(E)|^2}{\left|a_{j+1}(E)\left(a^0_p(E)-\rho^{-1}(E)\right)+b_{j+1}(E)a^0_{p-1}(E)\right|^2} = \frac{|a^0_{p-1}(E)|^2}{|v_{j+1}(E)|^2}$.

We rewrite $a^0_p(E) - \rho^{-1}(E) = a^0_p(E) - \cos(p\theta_p(E)) + i\sin(p\theta_p(E))$.

We put $\varphi(E) = a^0_p(E) - \cos(p\theta_p(E))$. We observe that $\varphi(E)$ is a C^1 function up to

E_0. On the other hand $\varphi(E_0) = 0$ since $a_p^0(E_0) - \rho^{-1}(E_0) = 0$. Hence, $\varphi(E) = (E - E_0)\,(\varphi'(E_0) + o(1))$. Note that $\sin\left(p\theta_p(E)\right) = c|E - E_0|^{1/2}(1 + o(1))$ with $c \neq 0$. Therefore, $a_p^0(E) - \rho^{-1}(E) = \tilde{c}|E - E_0|^{1/2}(1 + o(1))$ with $\tilde{c} \neq 0$.
On the other hand, $a_{j+1}(E_0) \neq 0$ and $a_{p-1}^0(E)$ has only simple roots. Hence, $\frac{|a_{p-1}^0(E)|^2}{|v_{j+1}(E)|^2} = t^2 g_5(t)$ where $g_5(t)$ is analytic near 0 and $g_5(0) \neq 0$.
Combining this with (6.1.22), there exist $C > 0$ independent of L and an analytic function $g_L(t)$ near 0, $g_L(0) \neq 0$ such that $q(E) = t^2 g_L(t)$ and $\max\{|g_L(t)|, |g_L'(t)|\} \leq C$ near 0.
To sum up, in both cases, $q(E) = t^2 g_L(t)$.
This implies directly that $a_k \asymp \frac{|\lambda_k - E_0|}{L-j}$ in the generic case. Moreover, $q'(E) = g_L(t) + \frac{t}{2} g_L'(t)$ where $E = E_0 + t^2$ and $t > 0$. Hence, $q(E)$, as a function of E, is C^1 up to E_0. Besides, its derivative near E_0 is bounded by a constant C_{lip} independent of L. As a result, (6.1.10) follows and we have the lemma proved. □

Remark 6.1.4. *Assume that the boundary point E_0 satisfies the condition $a_{p-1}^0(E_0) = a_{j+1}(E_0) = 0$. Then, by [Klo, Lemma 4.2] and the fact that the monodromy matrix $\widetilde{T_0}(E_0)$ defined in (5.1.3) is not diagonal, E_0 is an eigenvalue of H_L for all L large. Note that the hypothesis $a_{j+1}(E_0) = 0$ and (6.1.20) imply that $b_{j+1}(E_0) \neq 0$. Hence, $\frac{|a_{p-1}^0(E)|^2}{|v_{j+1}(E)|^2} = c(1 + o(1))$ near E_0 with $c \neq 0$. Combining this with (6.1.22), we infer that $a_k \asymp \frac{1}{L}$ for all eigenvalues λ_k of H_L close to E_0. This (non-generic) case will be treated in Chapter 7. Besides, it is not hard to check that, from the proof of Lemma 6.1.3, we find again the behavior of a_k stated in Remark 5.1.4 for both cases, the generic and non-generic one.*

Finally, we state and prove an asymptotic formula for eigenvalues of H_L close to a boundary point E_0.

Lemma 6.1.5. *Let $E_0 \in (-2, 2)$ be the left endpoint of the ith band $B_i = [E_0, E_1]$ of $\Sigma_{\mathbb{Z}}$. Let $\lambda_0^i < \lambda_1^i < \ldots < \lambda_{n_i}^i$ be eigenvalues of H_L in $\mathring{B}_i$, the interior of B_i.*
Pick $\varepsilon > 0$ a small, fixed number and $\varepsilon_1 \asymp \varepsilon^2$. Let $I = I_{\varepsilon_1} := [E_0, E_0 + \varepsilon_1] \subset (-2, 2) \cap \Sigma_{\mathbb{Z}}$. Assume that λ_k^i is an eigenvalue of H_L in I. Then, $k \leq \varepsilon L$ and

$$\lambda_k^i = E_0 + g\left(\frac{\pi(k+1)}{L}\right) + O\left(\frac{1}{L}\left(\frac{k}{L}\right)^3\right) \tag{6.1.23}$$

where g is a real analytic function near 0, $g(0) = g'(0) = 0$ and $g''(0) \neq 0$.
Consequently, for $\lambda_k^i \in \mathring{\Sigma}_{\mathbb{Z}}$ close to E_0, $\lambda_k^i - E_0 \asymp \frac{(k+1)^2}{L^2}$ (for $k \geq 1$, we will write $\lambda_k^i - E_0 \asymp \frac{k^2}{L^2}$ instead).

Moreover, there exists $\alpha > 0$ s.t. for any $k \neq n \leq \varepsilon L/C_1$ where $C_1 > 0$ is a large constant, we have

$$\frac{|k^2 - n^2|}{\alpha L^2} \leq |\lambda_k^i - \lambda_n^i| \leq \frac{\alpha |k^2 - n^2|}{L^2}. \tag{6.1.24}$$

Proof of Lemma 6.1.5. To simplify notations, we will skip the superscript i in λ_k^i of H_L throughout this proof.

First of all, from the property of θ_p and h_j near E_0, we have, for any E near E_0,

$$\theta_{p,L}(E) - \theta_{p,L}(E_0) = c(L)\sqrt{|E - E_0|}\,(1 + o(1)) \tag{6.1.25}$$

where $|c(L)|$ is lower bounded and upper bounded by positive constants independent of L. Put $L = Np + j$ where p is the period of the potential V and $0 \leq j \leq p-1$. According to Theorem 5.1.2, $\theta_{p,L}(E)$ is strictly monotone on B_i. W.o.l.g., we assume that $\theta_{p,L}(E)$ is strictly increasing on B_i. Note that, in this lemma, we enumerate eigenvalues λ_ℓ in $\mathring{B}_i$ with the index ℓ starting from 0. Then, we have to modify the quantization condition (5.1.7) in Theorem 5.1.2 appropriately. Recall that the quantization condition is $\theta_{p,L}(\lambda_\ell) = \frac{\pi\ell}{L-j}$ where $\frac{\pi\ell}{L-j} \in \theta_{p,L}(B_i)$ with $\ell \in \mathbb{Z}$. Assume that $\theta_p(E_0) = \frac{m\pi}{p}$ with $m \in \mathbb{Z}$. Put $\ell = \lambda N + \tilde{k}$ where $\lambda \in \mathbb{Z}$ and $0 \leq \tilde{k} \leq N-1$. We find $\lambda, \tilde{k}$ such that

$$\frac{\ell\pi}{Np} - \theta_{p,L}(E_0) = (\lambda - m)\frac{\pi}{p} + \frac{\tilde{k}\pi + h_j(E_0)}{Np} > 0. \tag{6.1.26}$$

It is easy to see that, for N large, the necessary condition is $\lambda - m \geq -1$. Consider the case $\lambda - m = -1$. Then, (6.1.26) yields

$$\tilde{k}\pi + h_j(E_0) > N\pi. \tag{6.1.27}$$

According to [Klo, Lemma 4.7], $h_j(E_0) \in \frac{\pi}{2}\mathbb{Z}$. We observe that if $h_j(E_0) < 0$, there does not exist $0 \leq \tilde{k} \leq N-1$ satisfying (6.1.27). Hence, $h_j(E_0) \in \frac{\pi}{2}\mathbb{N}$. We distinguish two cases. First of all, assume that $h_j(E_0) \in \pi\mathbb{N}$. Then, the first ℓ verifying (6.1.26) and $\lambda_\ell \in \mathring{\Sigma}_\mathbb{Z}$ is $\ell_0 = \frac{Np}{\pi}\theta_{p,L}(E_0) + 1$. Next, consider the case $h_j(E_0) \in \frac{\pi}{2} + \pi\mathbb{N}$. Then, the first ℓ chosen is $\ell_0 = \frac{Np}{\pi}\theta_{p,L}(E_0) + \frac{1}{2}$. Put $\ell_k = \ell_0 + k$ and we associate ℓ_k to λ_k, the $(k+1)-$th eigenvalue in $\mathring{B}_i$. Then, we always have

$$\theta_{p,L}(\lambda_k) - \theta_{p,L}(E_0) = \frac{(k+1)\pi}{L-j} + \frac{c_0}{L-j} \tag{6.1.28}$$

where $c_0 = 0$ if $h_j(E_0) \in \pi\mathbb{Z}$ and $c_0 = -\frac{\pi}{2}$ otherwise.

Hence, (6.1.25) and (6.1.28) yield $\lambda_k - E_0 \asymp \frac{(k+1)^2}{L^2}$ for all $\lambda_k \in I$ with ε small and L large.

Consequently, $k \lesssim \varepsilon L$ and $t_k = \lambda_k - E_0 \lesssim \left(\frac{k+1}{L-j}\right)^2$.
Next, we will find real analytic functions a, β and b such that

$$t_k := \lambda_k - E_0 = a(t_k)\beta^2 \left(\frac{\pi(k+1) + b(t_k)}{L-j}\right)$$

and $a(0) \neq 0$, $\beta(0) = 0$ and $\beta'(0) \neq 0$.
In other words, we would like to solve the following equation:

$$t = a(t)\beta^2 \left(\frac{\pi(k+1) + b(t)}{L-j}\right) \tag{6.1.29}$$

where $|t| \leq C\left(\frac{k+1}{L-j}\right)^2 \lesssim \varepsilon^2$.
First of all, we observe that the function $\varphi : t \mapsto \frac{t}{a(t)}$ is real analytic near 0, $\varphi(0) = 0$ and $\varphi'(0) = \frac{1}{a(0)} \neq 0$. Hence, the inverse function φ^{-1} is well defined in a neighborhood of 0. Now, by changing of variables $\tilde{t} := \varphi(t)$ and putting $\tilde{b} = b \circ \varphi$, the equation (6.1.29) reads

$$\tilde{t} = \beta^2 \left(\frac{\pi(k+1) + \tilde{b}(\tilde{t})}{L-j}\right) \tag{6.1.30}$$

Now we define $\psi_L(\tilde{t}) = \beta^2\left(\frac{\pi(k+1)+\tilde{b}(\tilde{t})}{L-j}\right) - \beta^2\left(\frac{\pi(k+1)+\tilde{b}(0)}{L-j}\right)$. Then, the equation (6.1.30) is equivalent to

$$\tilde{t} - \psi_L(\tilde{t}) = \beta^2 \left(\frac{\pi(k+1) + \tilde{b}(0)}{L-j}\right). \tag{6.1.31}$$

Note that $\psi_L(\tilde{t})$ is real analytic in a neighborhood of 0, $\psi_L(0) = 0$. Note that

$$\psi_L'(\tilde{t}) = 2\beta\left(\frac{\pi(k+1) + \tilde{b}(\tilde{t})}{L-j}\right) \cdot \beta'\left(\frac{\pi(k+1) + \tilde{b}(\tilde{t})}{L-j}\right) \cdot \frac{\tilde{b}'(\tilde{t})}{L-j}$$

and $\beta(0) = 0$. Hence, $|\psi_L'(\tilde{t})| \lesssim \frac{1}{L} \cdot \frac{(k+1)}{L} \lesssim \frac{\varepsilon}{L}$.
Consequently, $|(Id - \psi_L)'(0)| \geq \frac{1}{2}$ when L is large enough. This implies that (6.1.31) has a unique solution $\tilde{t}$ near 0. As a result, the solution of (6.1.29) is also unique in the interval $|t| \leq C\frac{(k+1)^2}{(L-j)^2}$.
Note that $\psi_L(0) = 0$, we obtain

$$\tilde{t} = \beta^2\left(\frac{\pi(k+1) + \tilde{b}(0)}{L-j}\right) + O\left(\frac{1}{L}\left(\frac{k}{L}\right)^3\right).$$

Therefore,

$$t = g\left(\frac{\pi(k+1)}{L}\right) + O\left(\frac{1}{L}\left(\frac{k}{L}\right)^3\right) \tag{6.1.32}$$

where $g = \varphi^{-1} \circ \beta^2$. Observe that g is real analytic around 0 and $g(0) = g'(0) = 0$. Moreover, a direct computation implies that $g''(0) \neq 0$. Then, we can write $g(x) = cx^2(1 + x\gamma(x))$ with $c \neq 0$, γ analytic near 0 and

$$\frac{1}{c}(g(x) - g(y)) = x^2 - y^2 + (x^3 - y^3)\gamma(x) + y^3(\gamma(x) - \gamma(y)).$$

Consequently, when L is large, $|\lambda_k - \lambda_n| \asymp \frac{|k^2-n^2|}{L^2}$ for all $k \neq n \leq \varepsilon L/C_1$ where $C_1 > 0$ is a large constant.

□

Remark 6.1.6. *For L large, the average distance between two consecutive, distinct eigenvalues (the spacing) is $\frac{1}{L}$. Lemma 6.1.5 says that, the spacing between eigenvalues near $\partial\Sigma_{\mathbb{Z}}$ is much smaller, the distance between λ_k^i and $\lambda_{k+1}^i \in I = [E_0, E_0 + \varepsilon_1]$ where $\varepsilon_1 \asymp \varepsilon^2$ has magnitude $\frac{k+1}{L^2}$. This fact implies that the number of eigenvalues in the interval I is asymptotically equal to εL as $L \to +\infty$.*

6.2 Small imaginary part

First of all, we prove the following lemma which will be useful when we estimate the sum $S_L(E)$.

Lemma 6.2.1. *Pick $\eta > 0$ and $E_0 \in \partial\Sigma_{\mathbb{Z}}$. For $E \in J := [E_0, E_0 + \eta] + i\mathbb{R}$, we define $S_{out}(E) = \sum\limits_{|\lambda_k - E_0| > 2\eta} \frac{a_k}{\lambda_k - E}$ Then,*

$$|S_{out}(E)| \leq \frac{1}{\eta} \textit{ and } |ImS_{out}(E)| \leq \frac{|ImE|}{\eta^2} \tag{6.2.1}$$

and

$$0 < S'_{out}(E) \leq \frac{1}{\eta^2} \textit{ for all } E \in [E_0, E_0 + \eta]. \tag{6.2.2}$$

Proof of Lemma 6.2.1. Note that $|\lambda_k - E| > \eta$ for all $|\lambda_k - E_0| > 2\eta$ and $E \in J$. On the other hand, $\text{Im}S_{\text{out}}(E) = \text{Im}E \sum\limits_{|\lambda_k - E_0| > 2\eta} \frac{a_k}{|\lambda_k - E|^2}$ and $S'_{\text{out}}(E) = \sum\limits_{|\lambda_k - E_0| > 2\eta} \frac{a_k}{(\lambda_k - E)^2}$. Hence, we have the claim follow. □

Now, we will prove that the imaginary part of $S_L(E) = \sum\limits_{k=0}^{L} \frac{a_k}{\lambda_k - E}$ is small if $|\text{Im}E|$ is not too small.

Lemma 6.2.2. *Assume the same hypothesis of the boundary point E_0 and we use the same enumeration for eigenvalues in the band B_i containing E_0 as in Lemma 6.1.5.*
Pick $\varepsilon > 0$ small, C_1, L large and $0 \leq n \leq \varepsilon L/C_1$. Consider the domain $\mathcal{A}_{n,\varepsilon} = [\frac{\lambda^i_{n-1}+\lambda^i_n}{2}, \frac{\lambda^i_n+\lambda^i_{n+1}}{2}] - i\left[C_0\frac{n+1}{L^2}, \varepsilon^5\right]$ where C_0 is a large constant and $\lambda^i_{-1} := 2E_0 - \lambda_0$. Then, for all $E \in \mathcal{A}_{n,\varepsilon}$ with ε sufficiently small, we have

$$|ImS_L(E)| \lesssim \varepsilon. \tag{6.2.3}$$

As a result, there are no resonances in $\mathcal{A}_{n,\varepsilon}$.

Proof of Lemma 6.2.2. Let $E = x - iy \in \mathcal{A}_{n,\varepsilon}$ with $y \in \left[C_0\frac{n}{L^2}, \varepsilon^5\right]$ and $x \in \left[\frac{\lambda^i_{n-1}+\lambda^i_n}{2}, \frac{\lambda^i_n+\lambda^i_{n+1}}{2}\right]$. Since Lemma 6.1.5, we can choose C_1 large enough so that $\lambda^i_n \leq E_0 + \varepsilon_1$ for all $n \leq \varepsilon L/C_1$ and $\lambda^i_k > E_0 + 2\varepsilon_1$ if $k > \varepsilon L$ and $\lambda^i_k \in B_i$ where $\varepsilon_1 \asymp \varepsilon^2$. Hence, Lemma 6.2.1 yields that

$$|\mathrm{Im}S_L(E)| \leq \frac{a^i_n y}{(\lambda^i_n - x)^2 + y^2} + \sum_{\substack{k=0\\k\neq n}}^{\varepsilon L} \frac{a^i_k y}{(\lambda^i_k - x)^2 + y^2} + \varepsilon \tag{6.2.4}$$

where $\{a^i_k\}$ are $\{a_k\}$ renumbered w.r.t. the band B_i. For the sake of simplicity, we will skip the superscript i in λ^i_k and a^i_k throughout the rest of proof.
Note that $\lambda_k \asymp \frac{(k+1)^2}{L^2}$ and $a_k \asymp \frac{(k+1)^2}{L^3}$ for all $0 \leq k \leq \varepsilon L$. Hence,

$$\frac{a_n y}{(\lambda_n - x)^2 + y^2} \leq \frac{a_n}{y} \lesssim \frac{n+1}{L} \lesssim \varepsilon.$$

Hence, it suffices to show that the sum

$$S = \sum_{\substack{k=0\\k\neq n}}^{\varepsilon L} \frac{a_k y}{(\lambda_k - x)^2 + y^2} \lesssim \varepsilon. \tag{6.2.5}$$

To simplify our notations, from now on, we will not write $0 \leq k \leq \varepsilon L$ in the sum. We upper bound S as follows

$$S \leq \sum_{\substack{k\neq n,\\|\lambda_k - x|\leq y}} \frac{a_k}{y} + \sum_{\substack{k\neq n,\\|\lambda_k - x|\geq y}} \frac{a_k y}{(\lambda_k - x)^2} = S_1 + S_2. \tag{6.2.6}$$

We will estimate S_1 first. For any index k of the sum S_1, we have, for C_0 sufficiently large,

$$|\lambda_k - \lambda_n| \leq |\lambda_k - x| + |x - \lambda_n| \leq 2y.$$

Hence, $|k^2 - n^2| \le 2CL^2y$ for some $C > 0$. In the other words,

$$(n^2 - 2CL^2y)_+ \le k^2 \le n^2 + 2CL^2y$$

with $(x)_+ = \max\{x, 0\}$. Hence,

$$\begin{aligned} S_1 &\lesssim \frac{1}{L^3y} \sum_{k=\sqrt{(n^2-2CL^2y)_+}}^{\sqrt{n^2+2CL^2y}} k^2 \lesssim \frac{1}{L^3y} \int_{\sqrt{(n^2-2CL^2y)_+}}^{\sqrt{n^2+2CL^2y}} x^2 dx \\ &\lesssim \frac{1}{L^3y}\left((n^2+2CL^2y)^{3/2} - (n^2-2CL^2y)_+^{3/2}\right). \end{aligned} \tag{6.2.7}$$

If $n^2 \le 2CL^2y$, the estimate (6.2.7) yields that

$$S_1 \lesssim \frac{1}{L^3y}(L^2y)^{3/2} \lesssim \sqrt{y}.$$

Otherwise,

$$S_1 \lesssim \frac{1}{L^3y} \frac{(n^2+2CL^2y)^3 - (n^2-2CL^2y)^3}{(n^2+2CL^2y)^{3/2} + (n^2-2CL^2y)^{3/2}} \lesssim \frac{1}{L^3y}\frac{L^2yn^4}{n^3} \lesssim \frac{n}{L}.$$

To sum up,

$$S_1 \lesssim \sqrt{y} + \frac{n}{L} \lesssim \varepsilon. \tag{6.2.8}$$

Now, we will find a good upper bound for S_2. Repeating the argument as above, we only need to consider the sum w.r.t. to indices $k^2 \ge n^2 + \frac{1}{C}yL^2$ or $k^2 \le n^2 - \frac{1}{C}yL^2$. We split and upper bound S_2 by two sums S_3 and S_4 which correspond to these two possibilities of index k.

$$\begin{aligned} S_3 &= \sum_{k=\sqrt{n^2+\frac{1}{C}yL^2}}^{\varepsilon L} \frac{a_k y}{(\lambda_k - x)^2} \lesssim \sum_{k=\sqrt{n^2+\frac{1}{C}yL^2}}^{\varepsilon L} \frac{a_k y}{(\lambda_k - \lambda_n)^2} \\ &\lesssim yL \sum_{k=\sqrt{n^2+\frac{1}{C}yL^2}}^{\varepsilon L} \frac{k^2}{(k-n)^2(k+n)^2} \lesssim yL \sum_{k=\sqrt{n^2+\frac{1}{C}yL^2}}^{\varepsilon L} \frac{1}{(k-n)^2} \\ &\lesssim yL \sum_{k=\sqrt{n^2+\frac{1}{C}yL^2}-n}^{\varepsilon L - n} \frac{1}{k^2} \lesssim yL \frac{1}{\sqrt{n^2+\frac{1}{C}yL^2} - n} \lesssim y \frac{1}{\sqrt{\left(\frac{n}{L}\right)^2 + \frac{1}{C}y} - \frac{n}{L}}. \end{aligned}$$

Note that, for all $a, b > 0$, $\frac{1}{\sqrt{a^2+b}-a} < \sqrt{b}$. Hence,

$$S_3 \lesssim y \cdot \frac{\sqrt{y}}{\sqrt{C}} \le \varepsilon. \tag{6.2.9}$$

Finally, we will estimate S_4. We only need to consider the case $y \le C\frac{n^2}{L^2}$. Then,

$$S_4 = \sum_{k=0}^{\sqrt{n^2-\frac{1}{C}yL^2}} \frac{a_k y}{(\lambda_k - x)^2} \lesssim \sum_{k=0}^{\sqrt{n^2-\frac{1}{C}yL^2}} \frac{a_k y}{(\lambda_k - \lambda_n)^2}$$
$$\lesssim yL \sum_{k=0}^{\sqrt{n^2-\frac{1}{C}yL^2}} \frac{k^2}{(n-k)^2(n+k)^2} \lesssim yL \sum_{n-\sqrt{n^2-\frac{1}{C}yL^2}}^{n} \frac{1}{k^2}.$$

Note that $n - \sqrt{n^2 - \frac{1}{C}yL^2} = \frac{\frac{1}{C}yL^2}{n+\sqrt{n^2-\frac{1}{C}yL^2}} \ge \frac{yL^2}{2Cn}$. Hence,

$$S_4 \lesssim yL \sum_{\frac{yL^2}{2Cn}}^{n} \frac{1}{k^2} \lesssim yL \cdot \frac{n}{yL^2} \lesssim \varepsilon. \tag{6.2.10}$$

From (6.2.6) and (6.2.8)-(6.2.10), the estimate (6.2.5), hence, (6.2.3) follows.
Note that, since $E_0 \in (-2, 2)$, $\theta(E) = \arccos \frac{E}{2}$ is analytic and $|\sin(\mathrm{Re}\theta(E))| \gtrsim 1$ near E_0. Consequently, for any $E \in [E_0, E_0 + \varepsilon_1] - i[0, \varepsilon^5]$, there exists a constant $c_0 > 0$ such that

$$|\mathrm{Im} e^{-i\theta(E)}| = e^{\mathrm{Im}\theta(E)} |\sin(\mathrm{Re}\theta(E))| \ge c_0.$$

Hence, there are no resonances in $\mathcal{A}_{n,\varepsilon}$. □

6.3 Resonances closest to the real axis

In the present section, we will give a proof for Theorem 6.0.3 which describes the resonances closest to the real axis. To do so, we will apply Rouché's theorem to show the existence and uniqueness of resonances in each rectangle $\mathcal{M}_n = \left[\frac{\lambda^i_{n-1}+\lambda^i_n}{2}, \frac{\lambda^i_n+\lambda^i_{n+1}}{2}\right] - i\left[0, C_0\frac{n+1}{L^2}\right]$ for $0 \le n \le \varepsilon L/C_1$ with $C_0, C_1 > 0$ large. Next, we derive the asymptotic formulae for resonances.

Corresponding to the case $n = 0$, we will apply Rouché's theorem in the rectangle $\left[E_0 - \varepsilon, \frac{\lambda^i_0+\lambda^i_1}{2}\right] - i\left[0, \frac{C_0}{L^2}\right]$ instead of $\left[E_0, \frac{\lambda^i_0+\lambda^i_1}{2}\right] - i\left[0, \frac{C_0}{L^2}\right]$. Next, we will prove that the unique resonance z_0 in this rectangle stays close to λ^i_0 at a distance $\frac{1}{L^3}$. Consequently, there are no resonances in $[E_0 - \varepsilon, E_0] - i\left[0, C_0\frac{n+1}{L^2}\right]$ and z_0 belongs to $\left[E_0, \frac{\lambda^i_0+\lambda^i_1}{2}\right] - i\left[0, \frac{C_0}{L^2}\right]$. Such a result is needed to study resonances below $\mathbb{R}\backslash\Sigma_{\mathbb{N}}$ in Section 6.4.
For that purpose, in Lemmata 6.3.2 and 6.3.3, we will use a different convention for λ^i_{-1} from that in Theorem 6.0.3. Concretely, we put $\lambda^i_{-1} := 2(E_0 - \varepsilon) - \lambda^i_0$ instead of $2E_0 - \lambda^i_0$ in these lemmata.

Note that, by Lemma 6.2.1, when we study the resonance equation near a boundary point E_0, only eigenvalues inside the spectrum and near E_0 need taking into account. In order to simplify the notation and the presentation, we will prove our results for $E_0 = \inf \Sigma_{\mathbb{Z}}$. For an arbitrary $E_0 \in \partial\Sigma_{\mathbb{Z}}$, all proofs work with tiny modifications. Note that when $E_0 = \inf \Sigma_{\mathbb{Z}}$ and if we ignore eigenvalues of H_L outside $\Sigma_{\mathbb{Z}}$, on the band containing E_0, two enumerations of eigenvalues (a usual one with increasing order and the other w.r.t. to bands of $\Sigma_{\mathbb{Z}}$) are the same. From now on, we will skip the superscript i in λ_k^i, a_k^i and the sum $S_{n,L}^i(E)$ defined in Theorem 6.0.3 can be written simply as

$$S_{n,L}(E) = \sum_{\substack{k=0 \\ k\neq n}}^{L} \frac{a_k}{\lambda_k - E}. \tag{6.3.1}$$

In order to use Rouché's theorem, we will need two following useful lemmata. Lemma 6.3.1 gives us an estimate on the sum $S_{n,L}(\lambda_n)$ and Lemma 6.3.2 show that, in $\mathcal{M}_n$, $S_{n,L}(E)$ can be approximated by $S_{n,L}(\lambda_n)$ with a small error.
Let's take a look at the sum $S_{n,L}(\lambda_n)$. First of all, the part of the sum w.r.t. $k > \varepsilon L$ is bounded by a constant depending only on ε. Next, from the asymptotic of a_k and λ_k near $\partial\Sigma_{\mathbb{Z}}$, it is easy to check that, in the absolute value, the sums $\sum\limits_{k=0}^{n-1} \frac{a_k}{\lambda_k - \lambda_n}$ and $\sum\limits_{k=n+1}^{\varepsilon L} \frac{a_k}{\lambda_k - \lambda_n}$ are of the same order $\frac{n \ln n}{L} \xrightarrow{L\to+\infty} +\infty$ if n is large ($n = \varepsilon L$ for example). However, we note that these two sums have opposite signs. We can actually show that they will cancel each other out to become very small (smaller than ε up to a constant factor). To make such a cancellation effect appear, the results on the smoothness of spectral data near $\partial\Sigma_{\mathbb{Z}}$ in Section 6.1 will be needed.

Lemma 6.3.1. *Let $\varepsilon > 0$ small and $0 \leq n \leq \varepsilon L/C_1$ with C_1 large. For $E \in \mathbb{C}$, let $S_{n,L}(E)$ be defined as in* (6.3.1).

$$|S_{n,L}(\lambda_n)| \lesssim \frac{1}{\varepsilon^2}. \tag{6.3.2}$$

Proof of Lemma 6.3.1. First of all, since Lemma 6.1.5, we can choose C_1 chosen to be large, we can assume that $\lambda_n \leq E_0 + \varepsilon_1$ and $\lambda_k \geq E_0 + 2\varepsilon_1$ with some $\varepsilon_1 \asymp \varepsilon^2$ for all $k \geq \varepsilon L$. Hence, Lemma 6.2.1 yield

$$S_{n,L}(\lambda_n) \lesssim \left| \sum_{\substack{k=0 \\ k\neq n}}^{2n} \frac{a_k}{\lambda_k - \lambda_n} \right| + \sum_{k=2n+1}^{\varepsilon L} \frac{a_k}{\lambda_k - \lambda_n} + \frac{1}{\varepsilon^2}. \tag{6.3.3}$$

Next, we estimate the sum $T = \sum\limits_{k=2n+1}^{\varepsilon L} \frac{a_k}{\lambda_k - \lambda_n}$. Recall that $|\lambda_n - \lambda_k| \asymp \frac{|k^2-n^2|}{L^2}$ and $a_k \asymp \frac{k^2}{L^3}$. Hence,

$$\begin{aligned}T &\asymp \frac{1}{L}\sum_{k=2n+1}^{\varepsilon L}\frac{k^2}{k^2-n^2} \asymp \frac{1}{L}(\varepsilon L - 2n - 1) + \frac{n^2}{L}\sum_{2n+1}^{\varepsilon L}\frac{1}{(k-n)(k+n)}\\ &\asymp \varepsilon + \frac{n}{L}\sum_{k=2n+1}^{\varepsilon L}\left(\frac{1}{k-n} - \frac{1}{k+n}\right) \asymp \varepsilon + \frac{n}{L}\left(\sum_{k=n}^{3n}\frac{1}{k} - \sum_{k=\varepsilon L - n}^{\varepsilon L + n}\frac{1}{k}\right)\\ &\asymp \varepsilon + \frac{n}{L} \asymp \varepsilon.\end{aligned} \tag{6.3.4}$$

Now we will show that $\left|\sum\limits_{\substack{k=0\\k\neq n}}^{2n} \frac{a_k}{\lambda_k - \lambda_n}\right| \lesssim \varepsilon$. We rewrite

$$S = \sum_{\substack{k=0\\k\neq n}}^{2n}\frac{a_k - a_n}{\lambda_k - \lambda_n} + a_n\sum_{\substack{k=0\\k\neq n}}^{2n}\frac{1}{\lambda_k - \lambda_n} = S_1 + S_2.$$

First, we will estimate S_1. Thanks to Lemma 6.1.3, we have

$$|S_1| \leq 2C_{\text{lip}} \cdot \frac{n}{L} \lesssim \varepsilon. \tag{6.3.5}$$

Second, we consider the sum S_2.

$$S_2 = a_n\left(\sum_{k=0}^{n-1}\left(\frac{1}{\lambda_k - \lambda_n} + \frac{1}{\lambda_{2n-k} - \lambda_n}\right)\right) = a_n\sum_{k=0}^{n-1}\frac{\lambda_k + \lambda_{2n-k} - 2\lambda_n}{(\lambda_k - \lambda_n)(\lambda_{2n-k} - \lambda_n)}. \tag{6.3.6}$$

Assume that $L = Np + j$ with $0 \leq j \leq p-1$. By Lemma 6.1.1, for each $k \leq 0 \leq n-1$, we can write $\lambda_k = \psi\left(\frac{k}{L}\right)$ where $\psi(x)$ is a C^2 function near E_0. Moreover, its second derivative near E_0 is bounded by a constant independent of L. Hence, we can apply the Taylor's expansion of the order 2 for the function $\psi(x)$ to get

$$\lambda_k + \lambda_{2n-k} - 2\lambda_n = \psi\left(\frac{k}{L}\right) + \psi\left(\frac{2n-k}{L}\right) - 2\psi\left(\frac{n}{L}\right) = O\left(\frac{(n-k)^2}{L^2}\right). \tag{6.3.7}$$

By (6.3.6) and (6.3.7), we infer that

$$\begin{aligned}|S_2| &\lesssim \frac{n^2}{L}\sum_{k=0}^{n-1}\frac{(n-k)^2}{(n^2-k^2)((2n-k)^2-n^2)} \lesssim \frac{n^2}{L}\sum_{k=0}^{n-1}\frac{1}{(n+k)(3n-k)}\\ &\lesssim \frac{n}{L}\left(\sum_{k=0}^{n-1}\frac{1}{n+k} + \sum_{k=0}^{n-1}\frac{1}{3n-k}\right) \lesssim \varepsilon.\end{aligned} \tag{6.3.8}$$

To sum up, thanks to (6.3.5) and (6.3.8), we have $|S_{n,L}(\lambda_n)| \lesssim \frac{1}{\varepsilon^2}$. □

For $E \in \mathcal{M}_n$, we compare $S_{n,L}(E)$ with $S_{n,L}(\lambda_n)$.

Lemma 6.3.2. *Pick C_1, C_0 large, $\varepsilon > 0$ small and $0 \le n \le \varepsilon L/C_1$. Let $\mathcal{M}_n = \left[\frac{\lambda_{n-1}+\lambda_n}{2}, \frac{\lambda_n+\lambda_{n+1}}{2}\right] - i\left[0, C_0\frac{n}{L^2}\right]$ and $S_{n,L}(E)$ be defined as in Lemma 6.3.1. We use the convention $\lambda_{-1} = 2(E_0 - \varepsilon) - \lambda_0$. Then, for all $E \in \mathcal{M}_n$, we have*

$$\left|S_{n,L}(E) - S_{n,L}(\lambda_n)\right| \lesssim L|E - \lambda_n| \lesssim \frac{n}{L}. \tag{6.3.9}$$

Proof of Lemma 6.3.2. By the definition of $S_{n,L}(E)$, we have

$$\left|S_{n,L}(E) - S_{n,L}(\lambda_n)\right| \le |\lambda_n - E| \sum_{k \ne n} \frac{a_k}{|\lambda_k - E||\lambda_k - \lambda_n|} \lesssim |\lambda_n - E| \sum_{k \ne n} \frac{a_k}{(\lambda_k - \lambda_n)^2}. \tag{6.3.10}$$

First of all, we observe that

$$S_1 = \sum_{k=0}^{n-1} \frac{a_k}{(\lambda_k - \lambda_n)^2} \lesssim L \sum_{k=0}^{n-1} \frac{(k+1)^2}{(n-k)^2(n+k)^2} \lesssim L \sum_{k=1}^{n} \frac{(n-k+1)^2}{k^2(2n-k)^2} \lesssim L. \tag{6.3.11}$$

Next,

$$S_2 = \sum_{k=n+1}^{\varepsilon L} \frac{a_k}{(\lambda_k - \lambda_n)^2} \lesssim L \sum_{k=n+1}^{\varepsilon L} \frac{k^2}{(k-n)^2(k+n)^2} \lesssim L \sum_{k=1}^{\varepsilon L - n} \frac{(k+n)^2}{k^2(k+2n)^2} \lesssim L. \tag{6.3.12}$$

Finally, put $S_3 = \sum_{k > \varepsilon L} \frac{a_k}{(\lambda_k - \lambda_n)^2}$. We can apply Lemma 6.2.1 to infer that S_3 is bounded by a constant depending only on ε. Combining this with (6.3.10)-(6.3.12), the claim follows. □

Now we will make use of the above lemma to show the existence and uniqueness of resonances in each rectangle $\mathcal{M}_n$ with $0 \le n \le \varepsilon L/C$.

Lemma 6.3.3. *Pick $C_1, C_0 > 0$ large, $\varepsilon > 0$ small and $0 \le n \le \varepsilon L/C_1$. Assume that $\mathcal{M}_n$ is the rectangle defined in Lemma 6.3.2 with the convention $\lambda_{-1} = 2(E_0 - \varepsilon) - \lambda_0$. Let $f(E) := S_L(E) + e^{-i\theta(E)}$ and $g_n(E) := \frac{a_n}{\lambda_n - E} + S_{n,L}(\lambda_n) + e^{-i\theta(\lambda_n)}$ where $S_{n,L}(E)$ is defined in (6.3.1). Then, f and g have the same number of zeros in $\mathcal{M}_n$. As a result, there is a unique resonance in $\mathcal{M}_n$.*

Proof of Lemma 6.3.3. Note that if λ_k is an eigenvalue of H_L which stays outside $\Sigma_{\mathbb{Z}}$, it is exponentially close to one of isolated simple eigenvalues of H_0^+ or H_j^- with $L = Np + j$ (see Theorem 5.1.2). Hence, we can choose ε to be sufficiently small such that $[E_0 - \varepsilon, E_0] \cap \sigma(H_L) = \emptyset$ for all L large.
Consequently, f and g are holomorphic in $\mathcal{M}_n$ for all $0 \le n \le \varepsilon L/C_1$. Recall that, in the present lemma, $\mathcal{M}_0 = \left[E_0 - \varepsilon, \frac{\lambda_0+\lambda_1}{2}\right] - i\left[0, C_0 \frac{n}{L^2}\right]$.
We will prove first that f and g have the same number of zeros in $\mathcal{M}_n$.
First of all, since Lemma 6.3.2, for all $E \in \mathcal{M}_n$, we have

$$\begin{aligned} |f(E) - g_n(E)| &\le \left| e^{-i\theta(E)} - e^{-i\theta(\lambda_n)} \right| + |S_{n,L}(E) - S_{n,L}(\lambda_n)| \\ &\lesssim L|\lambda_n - E| + \left| e^{-i\theta(E)} - e^{-i\theta(\mathrm{Re}E)} \right| + \left| e^{-i\theta(\mathrm{Re}E)} - e^{-i\theta(\lambda_n)} \right| \\ &\lesssim L|\lambda_n - E| + |\mathrm{Im}E| + |\mathrm{Re}E - \lambda_n| \lesssim L|\lambda_n - E| \lesssim \frac{n+1}{L}. \end{aligned} \tag{6.3.13}$$

Next we will check that, on the boundary $\gamma_n = ABCD$ of $\mathcal{M}_n$ (see Figure 6.1), $|g_n(E)|$ is much larger than $\frac{n}{L}$, hence, much larger than $|f(E) - g_n(E)|$.
To do so, we estimate the imaginary part of $g_n(E)$,

$$\begin{aligned} |\mathrm{Im} g_n(E)| &= \left| \frac{a_n \mathrm{Im}E}{(\lambda_n - \mathrm{Re}E)^2 + \mathrm{Im}^2 E} - e^{\mathrm{Im}\theta(\lambda_n)} \sin(\mathrm{Re}\theta(\lambda_n)) \right| \\ &\ge e^{\mathrm{Im}\theta(\lambda_n)} |\sin(\mathrm{Re}\theta(\lambda_n))| - \underbrace{\left| \frac{a_n \mathrm{Im}E}{(\lambda_n - \mathrm{Re}E)^2 + \mathrm{Im}^2 E} \right|}_{=P}. \end{aligned} \tag{6.3.14}$$

We will now upper bound P on the boundary $\gamma_n = ABCD$ of $\mathcal{M}_n$.
On the interval AB, E is real, hence, $P = 0$.
On AD, $\mathrm{Re}E = \frac{\lambda_{n-1}+\lambda_n}{2}$. Then,

$$(\lambda_n - \mathrm{Re}E)^2 + \mathrm{Im}^2 E \gtrsim (\lambda_n - \lambda_{n-1})^2 \gtrsim \frac{(n+1)^2}{L^4}.$$

The same bound holds for $E \in BC$.
Note that, when $n = 0$, $|\lambda_n - \lambda_{n-1}| = |\lambda_0 - (E_0 - \varepsilon)| \gtrsim \varepsilon \gg \frac{1}{L^2}$.
Finally, consider the interval CD with $|\mathrm{Im}E| = C_0 \frac{n+1}{L^2}$,

$$(\lambda_n - \mathrm{Re}E)^2 + \mathrm{Im}^2 E \ge \mathrm{Im}^2 E = C_0^2 \frac{(n+1)^2}{L^4}.$$

To sum up, on the curve γ_n,

$$P \lesssim \frac{a_n L^4 |\mathrm{Im}E|}{(n+1)^2} \lesssim \frac{n+1}{L}. \tag{6.3.15}$$

From (6.3.14)-(6.3.15), there exists a constant $c_0 > 0$ such that

$$|g_n(E)| \geq |\mathrm{Im} g_n(E)| \geq c_0 > |f(E) - g_n(E)| \text{ on } \gamma_n. \tag{6.3.16}$$

Then, (6.3.13), (6.3.16) and Rouché's theorem yield that f and g have the same number of zeros in $\mathcal{M}_n$.

We see that $g_n(E) = 0$ admits the unique solution $\tilde{z}_n$ in $\mathbb{C}$ given by:

$$\tilde{z}_n = \lambda_n + \frac{a_n}{S_{n,L}(\lambda_n) + e^{-i\theta(\lambda_n)}}. \tag{6.3.17}$$

Let's check that $\tilde{z}_n$ belongs to $\mathcal{M}_n$. Note that, by our convention for $\theta(E)$, $\theta(\lambda_n) \in [-\pi, 0]$, hence, $\mathrm{Im}\tilde{z}_n$ is negative. Moreover, since $|\sin(\theta(\lambda_n))| \geq c_0 > 0$, we have

$$|\lambda_n - \tilde{z}_n| \leq \frac{a_n}{|\sin(\theta(\lambda_n))|} \leq \frac{a_n}{c_0} \lesssim \frac{(n+1)^2}{L^3}. \tag{6.3.18}$$

Hence, $\tilde{z}_n \in \mathcal{M}_n$. This implies that the equation $f(E) = 0$ has a unique solution, say z_n, in $\mathcal{M}_n$ as well. In the other words, z_n is the unique resonance in $\mathcal{M}_n$. □

Finally, we complete the present chapter by giving a proof for the main theorem, Theorem 6.0.3.

Proof of Theorem 6.0.3. First of all, Lemmata 6.2.2 and 6.3.3 yield that there is one and only one resonance, say z_n, in each rectangle $B_{n,\varepsilon} = \left[\frac{\lambda_{n-1}+\lambda_n}{2}, \frac{\lambda_n+\lambda_{n+1}}{2}\right] - i[0, \varepsilon^5]$ for any $1 \leq n \leq \varepsilon L/C_1$. For $n = 0$, there is a unique resonance $z_0 \in \left[E_0 - \varepsilon, \frac{\lambda_0+\lambda_1}{2}\right] - i\left[0, \frac{C_0}{L^2}\right]$. Recall that we use the convention $\lambda_{-1} := 2E_0 - \lambda_0$ in Theorem 6.0.3. Then, $\frac{\lambda_{-1}+\lambda_0}{2}$ is equal to E_0, not $E_0 - \varepsilon$. We will prove later that z_0 actually stays inside the rectangle $\mathcal{M}_0 = \left[E_0, \frac{\lambda_0+\lambda_1}{2}\right] - i\left[0, \frac{C_0}{L^2}\right]$.

We will now take one step further to say something about the magnitude of z_n and its imaginary part. Let $\tilde{z}_n$ be the number defined in (6.3.17). Put $\alpha_n = S_{n,L}(\lambda_n) + e^{-i\theta(\lambda_n)}$. Then, since Lemma 6.3.1, $|\mathrm{Im}\alpha_n| = |\sin(\theta(\lambda_n))| \geq c_0 > 0$ and $|\alpha_n| \lesssim \frac{1}{\varepsilon^2}$.

Let's consider the square $D_{n,r} = \tilde{z}_n + r[-1,1]^2$ centered at $\tilde{z}_n$. We will choose $r < \frac{a_n}{|\alpha_n|}$ such that we can make sure that the resonance z_n belongs to $D_{n,r}$ by Rouché theorem. Precisely, we find r such that

$$|g_n(E)| > |f(E) - g_n(E)| \text{ on the boundary of } D_{n,r}. \tag{6.3.19}$$

First, we rewrite $g_n(E)$ as follows

$$|g_n(E)| = |g_n(E) - g_n(\tilde{z}_n)| = |E - \tilde{z}_n| \frac{a_n}{|\lambda_n - E||\lambda_n - \tilde{z}_n|} = |\alpha_n| \frac{|E - \tilde{z}_n|}{|\lambda_n - E|}. \tag{6.3.20}$$

Note that, for all $E \in \partial D_{n,r}$, $\frac{r}{2} \le |E - \tilde{z}_n| \le \frac{r}{\sqrt{2}}$. Combining this and (6.3.18), we infer that

$$|\lambda_n - E| \le |\lambda_n - \tilde{z}_n| + |E - \tilde{z}_n| \le \frac{a_n}{|\alpha_n|} + \frac{r}{\sqrt{2}} \le 2\frac{a_n}{|\alpha_n|}.$$

Hence,

$$|g_n(E)| \ge \frac{|\alpha_n|^2}{4a_n} r. \tag{6.3.21}$$

On the other hand, from (6.3.13), for all $E \in \partial D_{n,r}$,

$$|f(E) - g_n(E)| \lesssim L|\lambda_n - E| \lesssim L(|\lambda_n - \tilde{z}_n| + |\tilde{z}_n - E|) \lesssim L\frac{a_n}{|\alpha_n|}. \tag{6.3.22}$$

Hence, it suffices to choose $r < \frac{a_n}{|\alpha_n|}$ such that $\frac{|\alpha_n|^2}{4a_n} r \ge CL\frac{a_n}{|\alpha_n|}$ where C is a large constant. Obviously, $r = \frac{C}{|\alpha_n|^3} \cdot \frac{(n+1)^4}{L^5}$ satisfies the above inequality with C large.
Hence, by Rouché's theorem, the resonance z_n belongs to $D_{n,r}$ and

$$\left| z_n - \lambda_n - \frac{a_n}{\alpha_n} \right| \le \frac{C}{|\alpha_n|^3} \cdot \frac{(n+1)^4}{L^5}. \tag{6.3.23}$$

Hence, the asymptotic formula (6.0.1) follows.
We now estimate the imaginary part of z_n. Since (6.3.23), we have

$$\left| \mathrm{Im} z_n - \frac{a_n \sin(\theta(\lambda_n))}{|\alpha_n|^2} \right| \le \frac{C}{|\alpha_n|^3} \cdot \frac{(n+1)^4}{L^5}. \tag{6.3.24}$$

Consequently, the asymptotic formula (6.0.2) for $\mathrm{Im} z_n$ holds true and $|\mathrm{Im} z_n| \lesssim \frac{(n+1)^2}{L^3}$. Moreover, there exists $C > 0$ such that

$$|\mathrm{Im} z_n| \ge \frac{(n+1)^2}{|\alpha_n|^2 L^3} \left(\frac{1}{C} - \frac{C(n+1)^2}{|\alpha_n| L^2} \right) \gtrsim \varepsilon^4 \frac{(n+1)^2}{L^3}. \tag{6.3.25}$$

Finally, when $n = 0$, (6.3.23) yields $|z_0 - \lambda_0| \lesssim \frac{1}{L^3}$. Hence, z_0 belongs to the rectangle $\left[E_0, \frac{\lambda_0^i + \lambda_1^i}{2}\right] - i\left[0, \frac{C_0}{L^2}\right]$. On the other hand, z_0 is the unique resonance in the rectangle $\left[E_0 - \varepsilon, \frac{\lambda_0^i + \lambda_1^i}{2}\right] - i\left[0, \frac{C_0}{L^2}\right]$. As a result, there are no resonances in $[E_0 - \varepsilon, E_0] - i\left[0, \frac{C_0}{L^2}\right]$. We thus have Theorem 6.0.3 proved. □

6.4 Outside the spectrum

In the present section, we give a proof for Theorem 6.0.4 which describes a free resonance region of $H_L^{\mathbb{N}}$ below intervals which meet $\partial\Sigma_{\mathbb{Z}}$ from outside the spectrum $\Sigma_{\mathbb{Z}}$.

Proof of Theorem 6.0.4. First of all, by Theorem 6.0.3, there are no resonances in $[E_0-\varepsilon,E_0]-i\left[0,\frac{C_0}{L^2}\right]$ with $C_0>0$ large. Next, we will show that there are not resonances in $\mathcal{R}_1=[E_0-\varepsilon,E_0]-i\left[\frac{C_0}{L^2},\varepsilon^5\right]$ either. In order to do so, it suffices to prove that

$$|\mathrm{Im}S_L(E)|\lesssim\varepsilon \text{ in } \mathcal{R}_1. \tag{6.4.1}$$

Note that $S_L(E)$ is holomorphic in the domain $\mathring{\mathcal{R}}_1$. Hence, $|\mathrm{Im}S_L(E)|=-\mathrm{Im}S_L(E)$ is a harmonic function in $\mathring{\mathcal{R}}_1$. By the maximum principle for harmonic functions, it thus suffices to prove (6.4.1) on the boundary $\gamma=ABCD$ of $\mathcal{R}_1$ (see Figure 6.2).
Let $(\lambda^i_\ell)_\ell$ be (distinct) eigenvalues of H_L in the band B_i. Reasoning as in Lemma 6.2.2, we can assume that $\lambda^i_k>E_0+2\varepsilon^2$ for all $k>\varepsilon L$ and $\lambda^i_k\in B_i$ to get

$$|\mathrm{Im}S_L(E)|\lesssim\sum_{k=0}^{\varepsilon L}\frac{a^i_k y}{(\lambda^i_k-x)^2+y^2}+\varepsilon \text{ for all } z=x-iy\in\mathcal{R}_1 \text{ with } y>0. \tag{6.4.2}$$

Throughout the rest of the proof, we will skip the superscript i in λ^i_k and a^i_k.
From (6.4.2), it suffices to show that

$$S=\sum_{k=0}^{\varepsilon L}\frac{a_k y}{(\lambda_k-x)^2+y^2}\lesssim\varepsilon \text{ on } \gamma=ABCD. \tag{6.4.3}$$

First of all, we consider S on AB. On the interval AB, $y=\frac{C_0}{L^2}$ and $x\in[E_0-\varepsilon,E_0]$. Then, $|\lambda_k-x|\geq|\lambda_k-E_0|\gtrsim\frac{k^2}{L^2}\geq y$ for all $\varepsilon L\geq k\gtrsim\sqrt{C_0}$. Combining this with the fact that $a_k\asymp\frac{|\lambda_k-E_0|}{L}\asymp\frac{k^2}{L^3}$, we have

$$\sum_{\varepsilon L\geq k\geq\sqrt{C_0}}\frac{a_k y}{(\lambda_k-x)^2+y^2}\asymp\frac{1}{L^2}\sum_{\varepsilon L\geq k\geq\sqrt{C_0}}\frac{a_k}{(\lambda_k-x)^2}\lesssim\frac{1}{L^2}\sum_{\varepsilon L\geq k\geq\sqrt{C_0}}\frac{L}{k^2}\lesssim\frac{1}{L}. \tag{6.4.4}$$

On the other hand, we see that

$$\sum_{k=0}^{\sqrt{C_0}}\frac{a_k y}{(\lambda_k-x)^2+y^2}\leq\frac{1}{y}\sum_{k=0}^{\sqrt{C_0}}a_k\lesssim\frac{1}{L}\sum_{k=1}^{\sqrt{C_0}}k^2\lesssim\frac{1}{L}. \tag{6.4.5}$$

Therefore, (6.4.4)-(6.4.5) yield $S\lesssim\frac{1}{L}\ll\varepsilon$ on AB.
Next, on CD, $y=\varepsilon^5$ and $x\in[E_0-\varepsilon,E_0]$. We will split S into two sums S_1 and S_2. On the one hand, we estimate

$$S_1=\sum_{k=0}^{\varepsilon^2L-1}\frac{a_k y}{(\lambda_k-x)^2+y^2}\leq\frac{1}{y}\sum_{k=0}^{\varepsilon^2L-1}a_k\lesssim\frac{1}{\varepsilon^5L^3}\sum_{k=1}^{\varepsilon^2L-1}k^2\lesssim\frac{1}{\varepsilon^5L^3}\cdot(\varepsilon^2L)^3\lesssim\varepsilon. \tag{6.4.6}$$

On the other hand,

$$S_2 = \sum_{k=\varepsilon^2 L}^{\varepsilon L} \frac{a_k y}{(\lambda_k - x)^2 + y^2} \leq y \sum_{k=\varepsilon^2 L}^{\varepsilon L} \frac{a_k}{(\lambda_k - x)^2} \leq \varepsilon^5 L \sum_{k=\varepsilon^2 L}^{\varepsilon L} \frac{1}{k^2} \lesssim \varepsilon^5 L \left(\frac{1}{\varepsilon^2 L} - \frac{1}{\varepsilon L} \right) \lesssim \varepsilon^3. \tag{6.4.7}$$

By (6.4.6) and (6.4.7), we infer that $S \lesssim \varepsilon$ on CD.
Note that $|S| \lesssim \varepsilon$ on BC according to Lemma 6.2.2. Finally, we consider the sum S on the interval AD where $z = E_0 - \varepsilon - iy$ with $\frac{C_0}{L^2} \leq y \leq \varepsilon^5$. In this case, $|\lambda_k - x| \geq |x - E_0| - |\lambda_k - E_0| \gtrsim \varepsilon$ for all $0 \leq k \leq \varepsilon L$. Hence,

$$S \asymp y \sum_{k=0}^{\varepsilon L} \frac{a_k}{(\lambda_k - x)^2} \lesssim \frac{y}{\varepsilon^2} \sum_{k=0}^{\varepsilon L} a_k \lesssim \varepsilon^3 \sum_{k=1}^{\varepsilon L} \frac{k^2}{L^3} \lesssim \varepsilon^3 \cdot \frac{(\varepsilon L)^3}{L^3} \lesssim \varepsilon^6 < \varepsilon.$$

To sum up, S, hence $|\text{Im} S_L(E)|$, is bounded by ε up to a positive constant factor on $\mathcal{R}_1$. Therefore there are no resonances in $\mathcal{R}_1$ and the claim follows. □

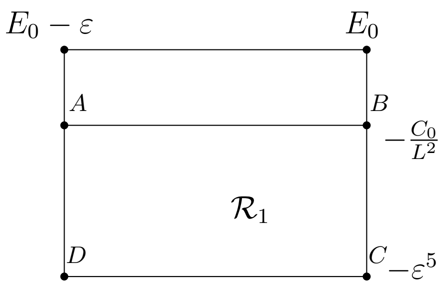


Figure 6.2: Free resonance region below $[E_0 - \varepsilon, E_0]$

CHAPTER 7

Resonances of $H_L^{\mathbb{N}}$ near $\partial\Sigma_{\mathbb{Z}}$ in the non-generic case

Contents

Let $E_0 \in (-2,2)$ be the left endpoint of the band B_i of $\Sigma_{\mathbb{Z}}$ and $L = Np + j$ with $0 \le j \le p-1$. In the present chapter, we consider the case that $a_k \asymp \frac{1}{L}$ for all eigenvalues $\lambda_k \in \mathring{\Sigma}_{\mathbb{Z}}$ close to E_0. As mentioned in the previous chapters, this case corresponds to either $E_0 \in \sigma(H_L)$ for L large or $a^0_{p-1}(E_0) \neq 0$ and $d_{j+1} = 0$ (see Remarks 5.1.4 and 6.1.4). We will study resonances in the domain $\mathcal{D} = [E_0, E_0 + \varepsilon_1] - i[0, \varepsilon_2]$ where $\varepsilon_1 \asymp \varepsilon^2$ and $\varepsilon_2 \asymp \varepsilon^5$ with $\varepsilon > 0$ small.

The asymptotic formula (6.1.23) leads us to make the rescaling $z = L^2(E - E_0)$ and track down rescaled resonances z in the new region $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_\varepsilon = [0, \varepsilon_1 L^2] - i[0, \varepsilon_2 L^2]$. Corresponding to this rescaling, we define rescaled eigenvalues $\tilde{\lambda}_k = L^2(\lambda_k - E_0)$ and $\tilde{a}_k = L a_k$. In the variable z, the resonance equation (5.1.1) is rewritten as

$$f_L(z) := \sum_{k=0}^{L} \frac{\tilde{a}_k}{\tilde{\lambda}_k - z} = -\frac{1}{L} e^{-i\theta(E)}. \tag{7.0.1}$$

Our goal is to describe solutions of (7.0.1) in the domain $\tilde{\mathcal{D}}$. Let $(\lambda^i_\ell)_\ell$ with $\ell \in [0, n_{i,\varepsilon}]$ be all (distinct) eigenvalues of H_L belonging to $[E_0, E_0 + \varepsilon_1] \subset B_i$. Note that $n_{i,\varepsilon} \asymp \varepsilon L$ for L large by Lemma 6.1.5. As in Chapter 6, we use the (local) enumeration w.r.t. bands of $\Sigma_{\mathbb{Z}}$ to enumerate eigenvalues in the band B_i. We renumber the corresponding a_k in the same way. Then, it suffices to study the rescaled resonance equation (7.0.1) in each rectangle $\mathcal{D}^i_n := [\tilde{\lambda}^i_n, \tilde{\lambda}^i_{n+1}] - i[0, \varepsilon^5 L^2]$ with $0 \le n \le \varepsilon L / C_1$ with $C_1 > 0$ large and in the rectangle $\mathcal{R}^i = [0, \tilde{\lambda}^i_0] - i[0, L^2 \varepsilon^5]$.

Lemma 6.1.5 implies that, for all $\lambda_k \in \mathring{B}_i$ and close to E_0, $|\tilde{\lambda}_k| \asymp (k+1)^2$ (we will use $|\tilde{\lambda}_k| \asymp k^2$ when $k \ge 1$). Moreover, from our assumption on a_k, the associated $\tilde{a}_k \asymp 1$. Note that, in the non-generic case, it is possible that $E_0 \in \sigma(H_L)$. Then, according to our enumeration, $\tilde{\lambda}^i_0 = 0$ and $\tilde{a}^i_0$ is still of order 1 (c.f. Remark 6.1.4).

In Section 7.1, we establish the subregions in $\mathcal{D}^i_n$ and $\mathcal{R}^i$ which contain no resonances. Next, in Section 7.2, we study the existence and uniqueness of resonances in the remaining subregions.

7.1 Resonance free regions

First of all, we state and prove the following lemma which will be useful for estimating $f_L(z)$.

Lemma 7.1.1. *Pick $\eta > 0$ and $E_0 \in \partial\Sigma_{\mathbb{Z}}$. For $E \in J := [E_0, E_0 + \eta] + i\mathbb{R}$, we define $z = L^2(E - E_0)$ and $f_{out}(z) = \sum_{|\lambda_k - E_0| > 2\eta} \frac{\tilde{a}_k}{\tilde{\lambda}_k - z}$ Then,*

$$|f_{out}(z)| \leq \frac{1}{\eta L} \text{ and } |Im f_{out}(z)| \leq \frac{|Im z|}{\eta^2 L^3} \tag{7.1.1}$$

and

$$0 < f'_{out}(z) \leq \frac{1}{\eta^2 L^3} \text{ for all } E \in [E_0, E_0 + \eta]\ . \tag{7.1.2}$$

Proof of Lemma 7.1.1. We put $S_{\text{out}}(E) = \sum_{|\lambda_k - E_0| > 2\eta} \frac{a_k}{\lambda_k - E}$. Then, the present lemma is a direct consequence of Lemma 6.2.1 and the fact that $f_{\text{out}}(z) = \frac{1}{L} S_{\text{out}}(E)$. □

7.1.1 Near the poles of $f_L(z)$

For each $0 \leq n \leq \varepsilon L / C_1$ with $C_1 > 0$ large, the rectangle $\mathcal{D}_n$ contains $\tilde{\lambda}_n, \tilde{\lambda}_{n+1}$, two poles of the meromorphic function $f_L(z)$. Since the modulus of $f_L(z)$ is big near these points, there are no resonances in those regions. Following is a quantitative version of this observation.

Lemma 7.1.2. *Let $E_0 \in (-2, 2)$ be the left endpoint of the band B_i of $\Sigma_{\mathbb{Z}}$. Assume that $(\lambda^i_\ell)_\ell$ with $0 \leq \ell \leq n_i$ are (distinct) eigenvalues of H_L in B_i. Put $I = [E_0, E_0 + \varepsilon_1] \subset B_i$ where $\varepsilon_1 \asymp \varepsilon^2$ with $\varepsilon > 0$ small. For each $0 \leq n \leq \varepsilon L / C_1$ with $C_1 > 0$ large, we define*

$$f_{n,L}(z) := \frac{\tilde{a}^i_n}{\tilde{\lambda}^i_n - z} + \frac{\tilde{a}^i_{n+1}}{\tilde{\lambda}^i_{n+1} - z}; \quad \tilde{f}_{n,L}(z) := f_L(z) - f_{n,L}(z) \tag{7.1.3}$$

where $z = L^2(E - E_0)$ with $E \in I - i[0, \varepsilon^5]$;
$\Delta_n := \frac{c_0(n+1)}{\kappa(\ln(n+1)+1)}$ where κ is a large constant.
Then,

- $|\tilde{f}_{n,L}(z)| \lesssim \frac{\ln(n+1)+1}{n+1}$ *for all $z \in [\tilde{\lambda}^i_n, \tilde{\lambda}^i_{n+1}] + i\mathbb{R}$,*
- $\tilde{f}'_{n,L}(z) \asymp \frac{1}{(n+1)^2}$ *if z is real and $z \in [\tilde{\lambda}^i_n, \tilde{\lambda}^i_{n+1}]$,*
- $|Im \tilde{f}_{n,L}(z)| \lesssim \frac{|Im z|}{n^2}$.

Consequently, for all $z \in \left([\tilde{\lambda}^i_n, \tilde{\lambda}^i_n \pm \Delta_n] \cap [0, \varepsilon_1 L^2]\right) - i[0, \Delta_n]$,

$$|f_L(z)| \gtrsim \frac{1}{\Delta_n} \gtrsim \frac{1}{\varepsilon L}. \tag{7.1.4}$$

Note that, in the definition of Δ_n, we choose κ to be large so that $\tilde{\lambda}^i_n - \Delta_n > 0$. Besides, $[\tilde{\lambda}^i_n, \tilde{\lambda}^i_n \pm \Delta_n]$ always belongs to $[0, \varepsilon_1 L^2]$ unless $n = 0$ and $\tilde{\lambda}^i_0 = 0$ i.e. $E_0 \in \sigma(H_L)$ for any L large.

Proof of Lemma 7.1.2. We can choose $C_1 > 0$ large enough such that $\lambda_n^i < E_0 + \varepsilon_1$ and $\lambda_k^i > E_0 + 2\varepsilon_1$ if $k > \varepsilon L$ and $\lambda_k^i \in B_i$. Then, Lemma 7.1.1 yield

$$\sum_{\lambda_k \notin [E_0, E_0+2\varepsilon_1]} \frac{\tilde{a}_k}{|\tilde{\lambda}_k - z|} \leq \frac{1}{\varepsilon_1 L} \lesssim \frac{\ln(n+1)+1}{n+1}. \tag{7.1.5}$$

Hence, it suffices to prove the same bound for the sum S where

$$S = \sum_{k=0}^{n-1} \frac{\tilde{a}_k^i}{\tilde{\lambda}_k^i - z} + \sum_{k=n+2}^{\varepsilon L} \frac{\tilde{a}_k^i}{\tilde{\lambda}_k^i - z} =: S_1 + S_2.$$

Throughout the rest of the proof, we will omit the superscript i to lighten the notation. Recall that, by Lemma 6.1.5, $|\tilde{\lambda}_k - \tilde{\lambda}_n| \asymp |k^2 - n^2|$ for all $k \neq n \in [0, \varepsilon L/C_1]$. Hence,

$$\begin{aligned} |S_1| &\leq \sum_{k=0}^{n-1} \frac{\tilde{a}_k}{|\tilde{\lambda}_k - z|} \leq \sum_{k=0}^{n-1} \frac{\tilde{a}_k}{|\tilde{\lambda}_n - \tilde{\lambda}_k|} \lesssim \sum_{k=0}^{n-1} \frac{1}{(n-k)(n+k)} \\ &\lesssim \sum_{k=1}^{n} \frac{1}{k(2n-k)} \lesssim \frac{\ln(n+1)+1}{n+1}. \end{aligned} \tag{7.1.6}$$

Next, we will estimate the sum S_2.

$$\begin{aligned} |S_2| &\leq \sum_{k=n+2}^{\varepsilon L} \frac{\tilde{a}_k}{|\tilde{\lambda}_k - z|} \lesssim \sum_{k \geq n+2} \frac{1}{k^2 - (n+1)^2} \lesssim \sum_{k \geq 1} \frac{1}{k(k+2n+2)} \\ &\lesssim \frac{1}{2n+2} \sum_{k=1}^{2n+2} \frac{1}{k} \lesssim \frac{\ln(n+1)+1}{n+1}. \end{aligned} \tag{7.1.7}$$

Hence, (7.1.5)-(7.1.7) yield $|\tilde{f}_{n,L}(z)| \lesssim \frac{\ln(n+1)+1}{n+1}$.
Now, we will prove the second item of Lemma 7.1.2. Assume that z is real and $z \in [\tilde{\lambda}_n, \tilde{\lambda}_{n+1}]$. Then, by Lemma 7.1.1, we have

$$\begin{aligned} \tilde{f}'_{n,L}(z) &\leq \sum_{\substack{k \leq \varepsilon L \\ k \neq n, n+1}} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - z)^2} + \frac{1}{\varepsilon_1^2 L^3} \\ &\leq \sum_{k=0}^{n-1} \frac{\tilde{a}_k}{(\tilde{\lambda}_n - \tilde{\lambda}_k)^2} + \sum_{k=n+2}^{\varepsilon L} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - \tilde{\lambda}_{n+1})^2} + \frac{1}{\varepsilon_1^2 L^3} \\ &\lesssim \frac{1}{(n+1)^2} + \frac{1}{\varepsilon_1^2 L^3} \lesssim \frac{1}{(n+1)^2}. \end{aligned} \tag{7.1.8}$$

On the other hand, for $z \in [\tilde{\lambda}_n, \tilde{\lambda}_{n+1}]$ and $n \geq 1$, we have

$$\begin{aligned} \tilde{f}'_{n,L}(z) &\geq \sum_{k=0}^{n-1} \frac{\tilde{a}_k}{(\tilde{\lambda}_n - \tilde{\lambda}_k)^2} \gtrsim \sum_{k=0}^{n-1} \frac{1}{(n-k)^2(n+k)^2} \\ &\gtrsim \sum_{k=1}^{n} \frac{1}{k^2(2n-k)^2} \gtrsim \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k^2} \gtrsim \frac{1}{n^2}. \end{aligned} \tag{7.1.9}$$

Moreover, if $z \in [\tilde{\lambda}_0, \tilde{\lambda}_1]$, it's easy to see that

$$\tilde{f}'_{n,L}(z) \geq \frac{\tilde{a}_2}{(\tilde{\lambda}_2 - \tilde{\lambda}_0)^2} \gtrsim 1. \tag{7.1.10}$$

Thanks to (7.1.8)-(7.1.10), we infer that that $\tilde{f}'_{n,L}(z) \asymp \frac{1}{(n+1)^2}$ for all $\tilde{\lambda}_n \leq z \leq \tilde{\lambda}_{n+1}$. Consequently, for $z \in [\tilde{\lambda}_n, \tilde{\lambda}_{n+1}] + i\mathbb{R}$,

$$|\mathrm{Im}\tilde{f}_{n,L}(z)| \leq |\mathrm{Im}z|\tilde{f}'_{n,L}(\mathrm{Re}z) \lesssim \frac{|\mathrm{Im}z|}{(n+1)^2}. \tag{7.1.11}$$

Finally, consider z which belongs to the square $[\tilde{\lambda}_n, \tilde{\lambda}_n + \Delta_n] - i[0, \Delta_n]$ or $[\tilde{\lambda}_{n+1} - \Delta_n, \tilde{\lambda}_{n+1}] - i[0, \Delta_n]$. W.o.l.g., assume that $z \in [\tilde{\lambda}_n, \tilde{\lambda}_n + \Delta_n] - i[0, \Delta_n]$.
Then, there exists $C > 0$ such that

$$|f_L(z)| \geq \frac{\tilde{a}_n}{|\tilde{\lambda}_n - z|} - \frac{\tilde{a}_{n+1}}{|\tilde{\lambda}_{n+1} - z|} - |\tilde{f}_{n,L}(z)| \geq \frac{1}{C\Delta_n} - \frac{C}{n} - \beta\frac{\ln n + 1}{n} \gtrsim \frac{1}{\Delta_n} \tag{7.1.12}$$

if the constant κ in the definition of Δ_n is chosen to be large. □

7.1.2 Large imaginary part

For each n, another region no containing resonances can be obtained from an estimate on $\mathrm{Im} f_L(z)$. Contrary to the generic case, when z is not too close to the real axis, $|\mathrm{Im} f_L(z)|$ becomes large instead of being small w.r.t. $\left|\frac{1}{L}\mathrm{Im}\left(e^{-i\theta(E)}\right)\right|$. Consequently, there are no resonances.

Lemma 7.1.3. *We assume the same hypotheses and notations in Lemma 7.1.2 and put* $x_0 := L^2(\lambda^i_{n+1} - \lambda^i_n) \asymp 2n + 1$.
Then, for $1 \leq n \leq \varepsilon L/C_1$, *we have* $|Imf_L(z)| \gtrsim \frac{1}{\varepsilon L}$ *for all* $\frac{x_0^2}{\varepsilon L} \leq |Imz| \leq \varepsilon^5 L^2$.
Besides, the above statement still holds in the region $[0, \tilde{\lambda}^i_1] - i\left[\frac{1}{\varepsilon L}, \varepsilon^5 L^2\right]$.

Proof of the Lemma 7.1.3. Throughout the proof, we will skip all superscript i in $\tilde{\lambda}_n^i, \tilde{a}_n^i$ associated to eigenvalues in B_i.
First of all, we have

$$|\mathrm{Im} f_L(z)| \geq \frac{\tilde{a}_n |\mathrm{Im} z|}{x^2 + |\mathrm{Im} z|^2} + \frac{\tilde{a}_{n+1}|\mathrm{Im} z|}{(x_0 - x)^2 + |\mathrm{Im} z|^2} + \sum_{\substack{k=0 \\ k \neq n, n+1}}^{\varepsilon L} \frac{\tilde{a}_k |\mathrm{Im} z|}{(\tilde{\lambda}_k - \mathrm{Re} z)^2 + |\mathrm{Im} z|^2} \tag{7.1.13}$$

where $x := \mathrm{Re} z - \tilde{\lambda}_n$.
Hence,

$$|\mathrm{Im} f_L(z)| \gtrsim \frac{|\mathrm{Im} z|}{x_0^2 + |\mathrm{Im} z|^2} \gtrsim \frac{1}{\varepsilon L} \cdot \frac{1}{1 + \frac{x_0^2}{\varepsilon^2 L^2}} \gtrsim \frac{1}{\varepsilon L} \tag{7.1.14}$$

for all $\frac{x_0^2}{\varepsilon L} \leq |\mathrm{Im} z| \leq \varepsilon L$.
Now, assume that $\varepsilon L \leq |\mathrm{Im} z| \leq \varepsilon^5 L^2$, we will find a good lower bound for the last term of RHS of (7.1.13). We compute

$$\begin{aligned}
A := & \sum_{\substack{k=0 \\ k \neq n, n+1}}^{\varepsilon L} \frac{\tilde{a}_k |\mathrm{Im} z|}{(\tilde{\lambda}_k - \mathrm{Re} z)^2 + |\mathrm{Im} z|^2} \\
= & \sum_{k=0}^{n-1} \frac{\tilde{a}_k |\mathrm{Im} z|}{(\tilde{\lambda}_k - \mathrm{Re} z)^2 + |\mathrm{Im} z|^2} + \sum_{k=n+2}^{\varepsilon L} \frac{\tilde{a}_k |\mathrm{Im} z|}{(\tilde{\lambda}_k - \mathrm{Re} z)^2 + |\mathrm{Im} z|^2} \\
\geq & \sum_{k=n+2}^{\varepsilon L} \frac{\tilde{a}_k |\mathrm{Im} z|}{|\mathrm{Im} z|^2 + C(k-n)^2(k+n)^2} \\
\gtrsim & \sum_{k=2}^{\varepsilon L/2} \frac{|\mathrm{Im} z|}{Ck^2(k+2n)^2 + |\mathrm{Im} z|^2} \gtrsim y^{1/2} \int_2^{\frac{1}{2}\varepsilon L} \frac{dt}{Ct^2(t+2n)^2 + y}
\end{aligned} \tag{7.1.15}$$

where $y := |\mathrm{Im} z|^2 \geq \varepsilon^2 L^2 \gg 1$.
Let's assume that $n \geq 1$. By the change of variables $t = y^{1/4} u$, we have

$$B := \int_2^{\frac{1}{2}\varepsilon L} \frac{dt}{t^2(t+2n)^2 + y} = y^{-3/4} \int_{2y^{-1/4}}^{\frac{1}{2}\varepsilon L y^{-1/4}} \frac{du}{Cu^2(u + 2ny^{-1/4})^2 + 1}.$$

Note that $\varepsilon L y^{-1/4} = \frac{\varepsilon L}{\sqrt{|\mathrm{Im} z|}} \geq \frac{\varepsilon L}{\varepsilon^2 L} = \frac{1}{\varepsilon}$ for all $|\mathrm{Im} z| \leq \varepsilon^5 L^2$. Hence, for $2\varepsilon < 10^{-3}$, we have

$$\begin{aligned}
A & \gtrsim y^{-1/4} \int_2^{1000} \frac{du}{u^2(u + 2ny^{-1/4})^2 + 1} \\
& \gtrsim \frac{1}{\sqrt{|\mathrm{Im} z|}} \int_2^{1000} \frac{du}{Cu^2(u + 2ny^{-1/4})^2 + 1}.
\end{aligned} \tag{7.1.16}$$

We observe that, if $\frac{n}{y^{1/4}} = \frac{n}{\sqrt{|\mathrm{Im} z|}}$ is smaller than a positive constant, say α i.e., $|\mathrm{Im} z| \geq n^2/\alpha$, the above integral is lower bounded by a positive constant C_α. Then,

$$A \gtrsim \frac{C_\alpha}{\sqrt{|\mathrm{Im} z|}} \gtrsim \frac{1}{\varepsilon^2 L} \text{ when } \frac{n^2}{\alpha} \leq |Imz| \leq \varepsilon^5 L^2.$$

Note that, if $\varepsilon L \geq \frac{n^2}{\alpha}$ i.e., $n \lesssim \sqrt{\varepsilon L}$, the above inequality holds true for all $\varepsilon L \leq |\mathrm{Im} z| \leq \varepsilon^5 L^2$.

Finally, we consider the case $n \gtrsim \sqrt{\varepsilon L}$ and find a lower bound for $|\mathrm{Im} f_L(z)|$ in the domain $\varepsilon L \leq |\mathrm{Im} z| \leq \frac{n^2}{\alpha}$ where α is a large, fixed constant.

Thanks to the first inequality in (7.1.15), we have

$$\begin{aligned} A &\gtrsim \sum_{k=0}^{n-1} \frac{|\mathrm{Im} z|}{(\tilde{\lambda}_k - \mathrm{Re} z)^2 + |\mathrm{Im} z|^2} \gtrsim \sum_{k=0}^{n-1} \frac{|\mathrm{Im} z|}{(\tilde{\lambda}_{n+1} - \tilde{\lambda}_k)^2 + |\mathrm{Im} z|^2} \\ &\gtrsim \sum_{k=0}^{n-1} \frac{|\mathrm{Im} z|}{(n+1-k)^2(n+1+k)^2 + |\mathrm{Im} z|^2} \\ &\gtrsim \sum_{k=2}^{n+1} \frac{|\mathrm{Im} z|}{(2n+2-k)^2 k^2 + |\mathrm{Im} z|^2} \gtrsim \sum_{k=2}^{n+1} \frac{|\mathrm{Im} z|}{n^2 k^2 + |\mathrm{Im} z|^2} \\ &= \frac{y_1}{n} \sum_{k=2}^{n+1} \frac{1}{k^2 + y_1^2} \gtrsim \frac{y_1}{n} \int_2^{n+2} \frac{dt}{t^2 + y_1^2} \end{aligned} \tag{7.1.17}$$

where $1 \leq \frac{\varepsilon L}{n} \leq y_1 := \frac{|\mathrm{Im} z|}{n} \leq \frac{n}{\alpha}$. Here, we choose $\alpha \geq 3$. Then, by the change of variables $t := y_1 u$, we have

$$A \gtrsim \frac{1}{n} \int_{2/y_1}^{(n+2)/y_1} \frac{du}{Cu^2 + 1} \gtrsim \frac{1}{n} \int_2^{\alpha} \frac{du}{u^2 + 1} \gtrsim \frac{1}{n} \gtrsim \frac{1}{\varepsilon L} \tag{7.1.18}$$

for all $\varepsilon L \leq |\mathrm{Im} z| \leq \frac{n^2}{\alpha}$.

Thanks to (7.1.16)-(7.1.18), we conclude that $|\mathrm{Im} f_L(z)| \geq \frac{C}{\varepsilon L}$ with $\varepsilon L \leq |\mathrm{Im} z| \leq \varepsilon^5 L^2$ for all $n \geq 1$.

Now, we consider the case $\mathrm{Re} z \in [0, \tilde{\lambda}_1]$. For all $1 \leq |\mathrm{Im} z| \leq \varepsilon^5 L^2$, we proceed as in

(7.1.15) and (7.1.16) to get

$$
\begin{aligned}
|\mathrm{Im} f_L(z)| &\geq c_0 \sum_{k=2}^{\varepsilon L} \frac{|\mathrm{Im} z|}{\alpha^2 k^4 + |\mathrm{Im} z|^2} \geq c_0 \sum_{k=2}^{\varepsilon L} \frac{|\mathrm{Im} z|}{\alpha^2 k^4 + |\mathrm{Im} z|^2} \\
&\geq \frac{c_0}{\sqrt{|\mathrm{Im} z|}} \int_{\frac{2}{\sqrt{|\mathrm{Im} z|}}}^{\frac{\varepsilon L}{\sqrt{|\mathrm{Im} z|}}} \frac{du}{\alpha^2 u^4 + 1} \geq \frac{c_0}{\sqrt{|\mathrm{Im} z|}} \int_2^{10000} \frac{du}{\alpha^2 u^4 + 1} \gtrsim \frac{1}{\varepsilon^2 L}.
\end{aligned} \tag{7.1.19}
$$

On the other hand, for $0 < |\mathrm{Im} z| < 1$, by putting $t := \frac{1}{\sqrt{|\mathrm{Im} z|}} \geq 1$, we have

$$
|\mathrm{Im} f_L(z)| \geq t \int_{2t}^{\varepsilon L t} \frac{du}{C u^4 + 1} \geq \frac{t}{C} \int_{2t}^{\varepsilon L t} \frac{du}{u^4} \gtrsim \frac{1}{t^2} \gtrsim |\mathrm{Im} z|. \tag{7.1.20}
$$

Thanks to (7.1.19) and (7.1.20), $|\mathrm{Im} f_L(z)| \gtrsim \dfrac{1}{\varepsilon L}$ for all $\frac{1}{\varepsilon L} \leq |\mathrm{Im} z| \leq \varepsilon^5 L^2$ and $\mathrm{Re} z \in [0, \tilde{\lambda}_1]$. Hence, the claim follows. □

We make a summary of obtained results on resonance free regions. Thanks to Lemmata 7.1.2 and 7.1.3, in $\mathcal{D}_n^i$ or $\mathcal{R}^i$, we obtain the resonance free regions greyed out in Figures 7.1-7.3. The white regions Ω_n^i, $\tilde{\Omega}_n^i$ and Ω^i in Figures 7.1-7.3 are the regions where we will track down resonances in Section 7.2.

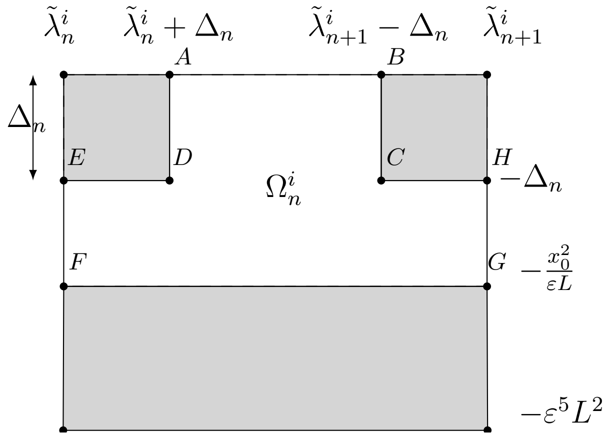


Figure 7.1: Resonance free region as $\Delta_n < \frac{x_0^2}{\varepsilon L}$

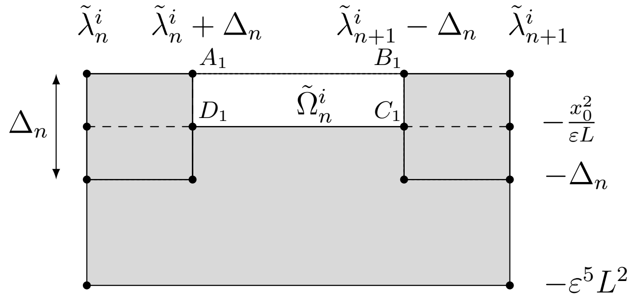


Figure 7.2: Resonance free resonance as $\Delta_n \geq \frac{x_0^2}{\varepsilon L}$

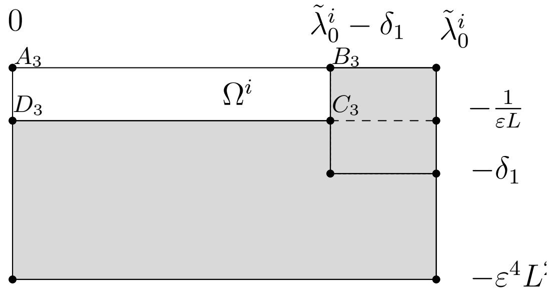


Figure 7.3: Resonance free region in $\mathcal{R}^i = [0, \tilde{\lambda}_0^i] - i[0, L^2\varepsilon]$

7.2 Resonances closest to the real axis

In the present section, for each band B_i of $\Sigma_{\mathbb{Z}}$, we will study rescaled resonances in $\Omega^i_n, \tilde{\Omega}^i_n$ and Ω^i (see Figures 7.1-7.3).

Convention: Recall that we use the (local) enumeration $(\lambda^i_\ell)_{\ell\geq 0}$ for (distinct) eigenvalues in the band $B_i \ni E_0$ and the usual enumeration $(\lambda_\ell)_\ell$ for eigenvalues of H_L outside the band B_i (written in increasing order and repeated according to their multiplicity). In the proofs of all results stated in this section, we will suppress the superscript i in $\tilde{\lambda}^i_k, \tilde{a}^i_k$ in order to lighten the notation. We will only specify the superscript i in case there is a risk of confusion. Note that, whenever we refer to λ_n, λ_{n+1} in this section, they are respectively $\lambda^i_n, \lambda^i_{n+1}$, the $(n+1)-$th and $(n+2)-$th eigenvalues in the band B_i. However, we will always use the notations λ_k or $\tilde{\lambda}_k$ to refer to the eigenvalues with the usual enumeration which does not depends on bands of $\Sigma_{\mathbb{Z}}$. Finally, as an abuse of notations, $\sum\limits_{k\neq n}$ and $\sum\limits_{k\neq n,n+1}$ stand for, respectively, $\sum\limits_{\lambda_k\neq\lambda^i_n}$ and $\sum\limits_{\lambda_k\neq\lambda^i_n,\lambda^i_{n+1}}$.

When $E_0 = \inf \Sigma_{\mathbb{Z}}$ and we ignore eigenvalues outsider $\Sigma_{\mathbb{Z}}$, two enumerations will be the same and readers can actually think of this case while following our proof.

7.2.1 Resonances in Ω^i_n

Recall that the region Ω^i_n corresponds to the case $\Delta_n < \frac{x_0^2}{\varepsilon L}$ with $x_0 = \tilde{\lambda}^i_{n+1} - \tilde{\lambda}^i_n$ which is equivalent to $\kappa(n+1)(\ln(n+1)+1) \gtrsim \varepsilon L$ (see Lemma 7.1.2 for the def. of Δ_n). Then, $n \geq \frac{\eta L}{\ln L}$ with some small $\eta \asymp \frac{\varepsilon}{\kappa}$.

The schema of studying resonances in Ω^i_n is split into two steps:

In Step 1, we will show that the number of solution of the resonance equation (7.0.1) is equal to that of the following equation by using Rouché's theorem.

$$f(z) := f_L(z) + \frac{1}{L}e^{-i\theta(E_0)} = 0 \tag{7.2.1}$$

Hence, we reduce our problem to count the number of solutions of (7.2.1). Note that, this number is exactly the cardinality of the set $f_L^{-1}\left(\left\{-\frac{1}{L}e^{-i\theta(E_0)}\right\}\right)$, the inverse image of the number $-\frac{1}{L}e^{-i\theta(E_0)}$.

Next, in Step 2, we partition Ω^i_n into two parts, the rectangles $ABCD$ and $EFGH$. First of all, we will show that the image of the boundary of the rectangle $ABCD$ under f_L is still a simple contour and on this contour, $|f'_L(z)| \gtrsim \frac{1}{n^2}$. Then, by the Argument Principal to the holomorphic function f_L in Ω^i_n, we infer that f_L is a conformal map from $ABCD$ onto $f_L(ABCD)$ and its inverse is holomorphic as well. Hence, there is at most one resonance

in this domain. The existence of that unique resonance depends on whether $f_L(ABCD)$ contains the point $-\frac{1}{L}e^{-i\theta(E_0)}$ or not.
Moreover, by studying $f_L(EFGH)$, we can conclude that there is at least one resonance which stays either in $ABCD$ or $EFGH$. Besides, if there is a resonance in $ABCD$, that will be the unique resonance in Ω_n^i.
Let's start the present subsection with the proof of the statement in Step 1:

Lemma 7.2.1. *The equations* (7.0.1) *and* (7.2.1) *have the same number of solutions in* Ω_n^i.

Proof of Lemma 7.2.1. Define $f(z)$ as in (7.2.1) and $g(z) := f_L(z) + \frac{1}{L}e^{-i\theta(E)}$.
First of all, we observe that f and g are holomorphic in $\overline{\Omega_n^i}$ since $\theta(E)$ is holomorphic in $[E_0, E_0+\varepsilon^2] + i\left[-\frac{x_0^2}{\varepsilon L^3}, 0\right]$ for all $E_0 \in (-2,2)$.
Moreover,

$$\begin{aligned}|f(z)-g(z)| &= \frac{1}{L}\left|e^{-i\theta(E)} - e^{-i\theta(E_0)}\right| \\ &\leq \frac{1}{L}\left|e^{-i\theta(E)} - e^{-i\theta(\mathrm{Re}E)}\right| + \frac{1}{L}\left|e^{-i\theta(\mathrm{Re}E)} - e^{-i\theta(E_0)}\right| \\ &\leq \frac{C}{L}|\mathrm{Im}E| + \frac{C}{L}|\mathrm{Re}E - E_0| \leq \frac{C}{L}\cdot\frac{n^2}{\varepsilon L^3} + \frac{C\varepsilon^2}{L} \leq \frac{C\varepsilon^2}{L}\end{aligned}$$

where the constant C is independent of ε.
Hence, to carry out the proof of the present lemma, it suffices to show that

$$|f(z)| \gtrsim \frac{1}{L} \text{ on the contour } \gamma_n = \partial\Omega_n^i.$$

Indeed, assume that we have such an estimate for $f(z)$ on γ_n. Then, $|f(z)-g(z)| \leq |f(z)|$ on γ_n. Hence, thanks to Rouché's theorem, f and g have the same number of zeros in the domain Ω_n^i.
Moreover, observe that $f_L(z)$ is real iff z is real. Hence, for $z \in \mathbb{R}$,

$$|f(z)| \geq |\mathrm{Im}f(z)| = \frac{1}{L}\left|\mathrm{Im}\left(e^{-i\theta(E_0)}\right)\right| = \frac{|\sin\theta(E_0)|}{L}.$$

Note that, $\sin(\theta(E_0)) \neq 0$ since $E_0 \in (-2,2)$.
Hence, to prove (7.2.1), it is sufficient to show that

$$|f_L(z)| \gtrsim \frac{1}{\varepsilon L} \text{ on } \gamma_n\backslash\mathbb{R}. \tag{7.2.2}$$

We decompose the contour γ_n into horizontal and vertical line segments as in Figure 7.1 . First of all, on the segments AD, BC, DE, CH, $|f_L(z)|$ is big (the zone near poles $\tilde{\lambda}_n$ and $\tilde{\lambda}_{n+1}$ of $f_L(z)$). More precisely, according to Lemma 7.1.2, on these segments,

$$|f_L(z)| \gtrsim \frac{1}{\Delta_n} \gtrsim \kappa \frac{\ln(n+1)+1}{n+1} \gtrsim \frac{1}{\varepsilon L}. \tag{7.2.3}$$

Next, on the segment FG, by Lemma 7.1.3, we have

$$|\text{Im} f_L(z)| \gtrsim \frac{1}{\varepsilon L}. \tag{7.2.4}$$

Finally, we study $f_L(z)$ on EF and GH. It suffices to consider the segment EF as $\tilde{\lambda}_n$ and $\tilde{\lambda}_{n+1}$ play equivalent roles.
Let $z \in EF$, hence, $z = \tilde{\lambda}_n - it$ with $\Delta_n \le t \le \frac{x_0^2}{\varepsilon L}$.
Then,

$$|\text{Im} f_L(z)| \ge |\text{Im} f_{n,L}(z)| = \frac{\tilde{a}_n}{t} + \frac{\tilde{a}_{n+1} t}{(\tilde{\lambda}_{n+1} - \tilde{\lambda}_n)^2 + t^2} \gtrsim \varphi(t)$$

where $\varphi(t) := \frac{1}{t} + \frac{t}{(\tilde{\lambda}_{n+1} - \tilde{\lambda}_n)^2 + t^2}$.
It's easy to check that $\varphi'(t) = -\frac{1}{t^2} + \frac{(\tilde{\lambda}_{n+1} - \tilde{\lambda}_n)^2 - t^2}{\left[(\tilde{\lambda}_{n+1} - \tilde{\lambda}_n)^2 + t^2\right]^2} \le 0$ for all $t \ne 0$. Hence, $\varphi(t)$ is (strictly) decreasing in the interval $\left[\Delta_n, \frac{x_0^2}{\varepsilon L}\right]$. Therefore,

$$|\text{Im} f_L(z)| \gtrsim \varphi\left(\frac{x_0^2}{\varepsilon L}\right) \gtrsim \frac{\varepsilon L}{n^2} + \frac{1}{\varepsilon L\left(1 + \frac{n^2}{(\varepsilon L)^2}\right)} \gtrsim \frac{1}{\varepsilon L}. \tag{7.2.5}$$

Thanks to (7.2.3)-(7.2.5), the claim in (7.2.2) follows and we have Lemma 7.2.1 proved. □

Now, we describe the image of the rectangles $ABCD$ and $EFGH$. First of all, we consider the rectangle $ABCD$ which is closer to the real axis.

Lemma 7.2.2. *Let $ABCD$ be the rectangle $[\tilde{\lambda}_n^i + \Delta_n, \tilde{\lambda}_{n+1}^i - \Delta_n] + i[-\Delta_n, 0]$ and γ_n^1 be its boundary.*
Then, $f_L(\gamma_n^1)$ is a simple contour. Besides, we have $|f_L'(z)| \gtrsim \frac{1}{n^2}$ on γ_n^1.

Proof of Lemma 7.2.2. First of all, on the horizontal segment AB where $\tilde{\lambda}_n + \Delta_n \le z \le \tilde{\lambda}_{n+1} - \Delta_n$, f_L is real-valued and

$$f_L'(z) = \sum_k \frac{\tilde{a}_k}{(\tilde{\lambda}_k - z)^2} \gtrsim \frac{1}{(\tilde{\lambda}_{n+1} - \tilde{\lambda}_n)^2} \gtrsim \frac{1}{n^2} > 0.$$

Hence, $f_L(z)$ is strictly increasing on AB. Then, f_L is injective and it transforms AB into an interval $[m_-^1, m_+^1]$ in $\mathbb{R}$. Note that, since Lemma 7.1.2, we have

$$\begin{aligned} m_-^1 &= f_L(\tilde{\lambda}_n + \Delta_n) \\ &= -\frac{\tilde{a}_n}{\Delta_n} + \frac{\tilde{a}_{n+1}}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_n - \Delta_n} + \tilde{f}_{n,L}(\tilde{\lambda}_n + \Delta_n) \lesssim -\frac{1}{\Delta_n} < 0 \end{aligned} \tag{7.2.6}$$

if the constant κ in the definition of Δ_n is large enough.
Similarly, $m_+^1 = f_L(\tilde{\lambda}_{n+1} - \Delta_n) \gtrsim \frac{1}{\Delta_n} \gtrsim \frac{\ln L}{L}$ for all $n > \frac{\eta L}{\ln L}$.
Now, for $z = x + iy \in \mathbb{C}$, we have

$$\begin{aligned} f_L'(z) &= \sum_k \frac{\tilde{a}_k}{(\tilde{\lambda}_k - z)^2} = \sum_k \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x - iy)^2} \\ &= \sum_k \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2} \cdot \frac{1}{\left(1 - \frac{iy}{\tilde{\lambda}_k - x}\right)^2} = \sum_k \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2} \cdot \frac{\left(1 + \frac{iy}{\tilde{\lambda}_k - x}\right)^2}{\left[1 + \left(\frac{y}{\tilde{\lambda}_k - x}\right)^2\right]^2} \end{aligned} \tag{7.2.7}$$

Note that, for any holomorphic function f, $f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$ with $z = x + iy$. Hence, $\frac{\partial}{\partial x}\mathrm{Re} f_L(z) = \frac{\partial}{\partial y}\mathrm{Im} f_L(z) = \mathrm{Re}[f_L'(z)]$ and we put

$$p(x,y) := \mathrm{Re}[f_L'(z)] = \sum_k \underbrace{\frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2} \cdot \frac{1 - \left(\frac{y}{\tilde{\lambda}_k - x}\right)^2}{\left[1 + \left(\frac{y}{\tilde{\lambda}_k - x}\right)^2\right]^2}}_{=:p_k(x,y)}. \tag{7.2.8}$$

Next, we study f_L on the line segment AD where $z = x + iy$ with $x = \tilde{\lambda}_n + \Delta_n$ and $-\Delta_n \leq y \leq 0$. In the present case, the identity (7.2.8) reads

$$\mathrm{Re}[f_L'(z)] = \tilde{a}_n \frac{\Delta_n^2 - y^2}{(\Delta_n^2 + y^2)^2} + \sum_{k \neq n} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2} \cdot \frac{1 - \left(\frac{y}{\tilde{\lambda}_k - x}\right)^2}{\left[1 + \left(\frac{y}{\tilde{\lambda}_k - x}\right)^2\right]^2}. \tag{7.2.9}$$

along the segment AD.
Note that, for any $\lambda_k \neq \lambda_n^i$, we have $(\tilde{\lambda}_k - x)^2 \gtrsim n^2 \gg \Delta_n^2$ for all $z \in AD$. Recall that we are considering the case that $n \geq \eta \frac{L}{\ln L}$.
Hence, all the terms in RHS of (7.2.9) are positive for all $y \in [-\Delta_n, 0]$. This implies that

$$\frac{\partial}{\partial y}\mathrm{Im} f_L(z) = \mathrm{Re}[f_L'(z)] \gtrsim \sum_{k \neq n} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2} \gtrsim \frac{1}{n^2}.$$

Hence, along the segment AD, $\text{Im}f_L(z)$ is strictly increasing in $y = \text{Im}z$. As a result, $f_L(z)$ is injective on AD and

$$\begin{aligned} 0 &\geq \text{Im}f_L(z) \geq \text{Im}f_L(D) = \text{Im}f_L(\tilde{\lambda}_n + \Delta_n - i\Delta_n) \\ &= -\frac{\tilde{a}_n}{2\Delta_n} - \sum_{k\neq n} \frac{\tilde{a}_k \Delta_n}{(\tilde{\lambda}_k - \tilde{\lambda}_n)^2 + \Delta_n^2} \asymp -\frac{1}{\Delta_n} - \frac{\Delta_n}{n^2} \asymp -\frac{1}{\Delta_n}. \end{aligned} \tag{7.2.10}$$

Note that, on AD, the $n-$th term $\frac{\tilde{a}_n y}{y^2+\Delta_n^2}$ is much bigger than the sum of the other terms in $\text{Im}f_L(z)$. Precisely,

$$\text{Im}f_L(z) = \frac{\tilde{a}_n y}{y^2 + \Delta_n^2}\left(1 + O\left(\frac{1}{\kappa^2 \ln^2 n}\right)\right) = \frac{\tilde{a}_n y}{y^2 + \Delta_n^2}\left(1 + O\left(\frac{1}{\kappa^2 \ln^2 L}\right)\right).$$

Hence, the monotonicity of $\text{Im}f_L(z)$ in y on AD just comes from that of $\frac{\tilde{a}_n y}{y^2+\Delta_n^2}$.
Next, we will estimate $\text{Re}f_L(z)$ on AD. For all $-\Delta_n \leq y \leq 0$, by Lemma 7.1.2, we have

$$\begin{aligned} \text{Re}f_L(z) &= -\frac{\tilde{a}_n\Delta_n}{\Delta_n^2 + y^2} + \frac{\tilde{a}_k(\tilde{\lambda}_{n+1} - \tilde{\lambda}_n - \Delta_n)}{(\tilde{\lambda}_{n+1} - \tilde{\lambda}_n - \Delta_n)^2 + y^2} + \text{Re}\tilde{f}_{n,L}(z) \\ &\asymp -\frac{1}{\Delta_n} + O\left(\frac{\ln n}{n}\right). \end{aligned} \tag{7.2.11}$$

Hence, when we choose the constant κ in the definition of Δ_n to be big enough, $-\frac{\tilde{a}_n\Delta_n}{\Delta_n^2+y^2}$ becomes the dominating term in RHS of (7.2.11). Then, $\text{Re}f_L(z) \asymp -\frac{1}{\Delta_n}$.
By the equivalent role between $\tilde{\lambda}_n$ and $\tilde{\lambda}_{n+1}$, we obtain a similar result for the image of BC under f_L, that is, the $\text{Im}f_L(z)$ is increasing in $y = \text{Im}z \in [-\Delta_n, 0]$, $\text{Re}f_L(z) \asymp \frac{1}{\Delta_n}$ and $|f_L'(z)| \gtrsim \frac{1}{n^2}$.
Finally, we consider f_L on $CD = \{z = x - i\Delta_n | x \in [\tilde{\lambda}_n + \Delta_n, \tilde{\lambda}_{n+1} - \Delta_n]\}$.
Note that, in the present case, the function $p_k(x, y)$ in (7.2.8) is strictly positive for any $k \neq n, n+1$ and non-negative otherwise. Hence, $\text{Re}f_L(z)$ is strictly increasing in x. Hence, on CD,

$$-\frac{1}{\Delta_n} \asymp \text{Re}f_L(D) \leq \text{Re}f_L(z) \leq \text{Re}f_L(C) \asymp \frac{1}{\Delta_n}.$$

Moreover, we have the following estimate for $|f_L'(z)|$:

$$|f_L'(z)| \geq \text{Re}[f_L'(z)] \gtrsim \frac{1}{n^2} \cdot \frac{1 - \left(\frac{\Delta_n}{\tilde{\lambda}_{n+2} - x}\right)^2}{\left[1 + \left(\frac{\Delta_n}{\tilde{\lambda}_{n+2} - x}\right)^2\right]^2} \gtrsim \frac{1}{n^2}$$

for all $x \in [\tilde{\lambda}_n + \Delta_n, \tilde{\lambda}_{n+1} - \Delta_n]$.
Finally, we give estimates on $\text{Im} f_L(z)$ on CD. First of all, on this segment, we have

$$-\text{Im} f_{n,L}(z) = \frac{\tilde{a}_n \Delta_n}{(\tilde{\lambda}_n - x)^2 + \Delta_n^2} + \frac{\tilde{a}_{n+1} \Delta_n}{(\tilde{\lambda}_{n+1} - x)^2 + \Delta_n^2}.$$

It's easy to see that, as x varies in $[\tilde{\lambda}_n + \Delta_n, \tilde{\lambda}_{n+1} - \Delta_n]$, $\frac{\Delta_n}{n^2} \lesssim -\text{Im} f_{n,L}(z) \lesssim \frac{1}{\Delta_n}$.
On the other hand, for $x \in [\tilde{\lambda}_n + \Delta_n, \tilde{\lambda}_{n+1} - \Delta_n]$,

$$-\text{Im} \tilde{f}_{n,L}(z) = \Delta_n \sum_{k \neq n, n+1} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2 + \Delta_n^2} \asymp \Delta_n \sum_{k \neq n, n+1} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2} \asymp \frac{\Delta_n}{n^2}.$$

Note that, $\frac{1}{\Delta_n} \gg \frac{\Delta_n}{n^2}$. Hence, $\text{Im} f_L(z)$ varies from $-\frac{1}{\Delta_n}$ to $-\frac{\Delta_n}{n^2}$ on the CD.
To sum up, the holomorphic function f_L is injective on each edge of the rectangle $ABCD$. Hence, the image of each edge under f_L is a non self-intersecting continuous curve. Obviously, since $f_L(AB)$ is a segment in the real axis, it does not intersect the other curves. It's easy to see that $f_L(AD) \cap f_L(BC) = \emptyset$ as well. However, it's not so evident that $f_L(AD)$ and $f_L(CD)$ only intersect at $f_L(D)$. In order to prove that, it is necessary to use the estimate on the derivative of $f_L(z)$. Note that $|f_L'(z)| \gtrsim \frac{1}{n^2}$ on all edges of $ABCD$. On the other hand, f_L is holomorphic in a neighborhood of the rectangle $ABCD$. Hence, f_L is locally bi-holomorphic near any point on AB, BC, CD, DA. Hence, $f_L(AD)$ only intersects $f_L(CD)$ at $f_L(D)$. Hence, $f_L(\gamma_n^1)$ is a simple contour and $|f_L'(z)| \gtrsim \frac{1}{n^2}$ on γ_n^1. □

Lemma 7.2.3. *Let $ABCD$ be the rectangle $[\tilde{\lambda}_n^i + \Delta_n, \tilde{\lambda}_{n+1}^i - \Delta_n] + i[-\Delta_n, 0]$. Then, the function $f_L(z)$ is a bijection from the rectangle $ABCD$ onto $f_L(ABCD)$ and its inverse in $A'B'C'D' = f_L(ABCD)$ is holomorphic as well. Moreover,*

$$|f_L'(z)| \gtrsim \frac{1}{n^2} \text{ for all } z \text{ belonging to the rectangle } ABCD.$$

Consequently, $f_L(z)$ is a conformal map in the interior of the rectangle $ABCD$, hence, the angles between boundary curves of $A'B'C'D'$ are all $90°$.

Proof of Lemma 7.2.3. Denote by γ_n^1 the boundary of the rectangle $ABCD$. According to Lemma 7.2.2, $f_L(\gamma_n^1)$ is a simple contour and f_L is injective in γ_n^1. Hence, for any interior point w of $f_L(ABCD)$, the contour $f_L(\gamma_n^1)$ travels counterclockwise around w exactly one times. Hence, thanks to Argument Principle, the equation $f_L(z) = w$ in $ABCD$ has the unique solution. In other words, f_L is injective in the interior of the rectangle $ABCD$. By using Open Mapping Theorem, we infer that $f_L(z)$ is bijective from the rectangle $ABCD$

onto $f_L(ABCD)$ and its inverse in $f_L(ABCD)$ is holomorphic. Moreover, $f_L'(z) \neq 0$ for all z in the rectangle $ABCD$. Hence, by using the Maximum Modulus Principle for the holomorphic function $\frac{1}{f_L'}$, we have $|f_L'(z)| \gtrsim \frac{1}{n^2}$ in $ABCD$.
The holomorphic function f_L is therefore a conformal map in the rectangle $ABCD$ and the claim follows. □

We observe that the domain $MNOP \backslash A'B'C'D'$ is included in the image of $EFGH$ under f_L.

Lemma 7.2.4. *Put* $MNOP = \left[-\frac{1}{C\Delta_n}, \frac{1}{C\Delta_n}\right] - i\left[0, \frac{2|\sin\theta(E_0)|}{L}\right]$ *with* $C > 0$ *large. Let* $A'B'C'D' = f_L(ABCD)$.
Assume that $-\frac{1}{L}e^{-i\theta(E_0)} \in MNOP \backslash A'B'C'D'$. *Then,*

$$f_L(EFGH) \supset MNOP \backslash A'B'C'D'.$$

We will skip the proof of Lemma 7.2.4 for a while and make use of this lemma to describe resonances in the domain Ω_n^i.

Theorem 7.2.5. *Assume that* $n > \eta \frac{L}{\ln L}$ *and put* $x_0 = \tilde{\lambda}_{n+1}^i - \tilde{\lambda}_n^i$.
Let Ω_n^i *be the complement of two squares* $[\tilde{\lambda}_n^i, \tilde{\lambda}_n^i + \Delta_n] + i[-\Delta_n, 0]$ *and* $[\tilde{\lambda}_{n+1}^i, \tilde{\lambda}_{n+1}^i - \Delta_n] + i[-\Delta_n, 0]$ *in the rectangle* $[\tilde{\lambda}_n^i, \tilde{\lambda}_{n+1}^i] + i\left[-\frac{x_0^2}{\varepsilon L}, 0\right]$ *(the region ABCHGFED in Figure 7.1).*
Then, there exists at least one rescaled resonance in Ω_n^i. *Hence,* $|Imz| \lesssim \frac{n^2}{\varepsilon L}$ *for all resonances in* Ω_n^i.
Moreover, if $-\frac{1}{L}e^{-i\theta(E_0)}$ *belongs to* $A'B'C'D' = f_L(ABCD)$, *the rescaled resonance, says* z_n, *is unique and*

$$|Imz_n| \leq \Delta_n = \frac{n}{\kappa \ln n} \asymp \frac{n}{\kappa \ln L} \lesssim \frac{n^2}{\varepsilon L}.$$

Proof of Theorem 7.2.5. Recall that, thanks to Lemma 7.2.1, the number of rescaled resonances z is the cardinality of $f_L^{-1}\left(\left\{-\frac{1}{L}e^{-i\theta(E_0)}\right\}\right)$. Hence, for the existence of resonances, we have to check if the point $-\frac{1}{L}e^{-i\theta(E_0)}$ belongs to Ω_n^i. Note that $-\frac{1}{L}e^{-i\theta(E_0)}$ always stays inside the open rectangle $MNOP = \left[-\frac{1}{C\Delta_n}, \frac{1}{C\Delta_n}\right] - i\left[0, \frac{2|\sin\theta(E_0)|}{L}\right]$ where $C > 0$ is a big constant. We consider two possibilities. First of all, assume that $-\frac{1}{L}e^{-i\theta(E_0)}$ belongs to $A'B'C'D'$. Then, by Lemma 7.2.3, there exists one and only one rescaled resonance z_n in $ABCD$. When that case happens, $|\mathrm{Im} z_n| \leq \Delta_n = \frac{n}{\kappa \ln n} \asymp \frac{n}{\kappa \ln L}$. Remark that, this case can not happen for all $\varepsilon L \geq n > \eta \frac{L}{\ln L}$. For example, when $n = \varepsilon L$ i.e., the real part of resonance is far from $\partial\Sigma_{\mathbb{Z}}$ by a constant distance, there are no rescaled resonances in

$ABCD$ from Theorem 5.2.4.
Now, assume that the other case happens i.e., $-\frac{1}{L}e^{-i\theta(E_0)} \in MNOP\backslash A'B'C'D'$.
In this case, $f_L(EFGH)$ contains $MNOP\backslash A'B'C'D'$ by Lemma 7.2.4. Then, $-\frac{1}{L}e^{-i\theta(E_0)}$ stays in the image of $EFGH$ under f_L. Hence, there exists a rescaled resonance in the rectangle $EFGH$ and note that the imaginary part of such a rescaled resonance is smaller than $-\frac{x_0^2}{\varepsilon L}$ and bigger than $-\Delta_n$. □

Finally, to complete the subsection, we state here the proof of Lemma 7.2.4.

Proof of Lemma 7.2.4. Note that $|\mathrm{Re} f_L(z)|$ is bigger than $\frac{1}{C\Delta_n}$ on segments AD and BC if C is large enough.
Then, the hypothesis that $-\frac{1}{L}e^{-i\theta(E_0)} \in MNOP\backslash A'B'C'D'$ yields $\frac{\Delta_n}{n^2} \leq 2\frac{|\sin(\theta(E_0))|}{L}$.
Hence,

$$\frac{\Delta_n}{n^2} \leq 2\frac{|\sin(\theta(E_0))|}{L} \leq \frac{1}{\varepsilon L} \ll \frac{1}{\Delta_n}.$$

By the open mapping theorem, the image of the open rectangle $EFGH$ is still a bounded domain in $\mathbb{C}$. From the study of the curve $f_L(CD)$ in Lemma 7.2.2, we know that, the imaginary part of $f_L(CD)$ increases from $-\frac{1}{\Delta_n}$ to $-\frac{\Delta_n}{n^2}$. Hence, it suffices to show that the imaginary part of f_L on all parts of the boundary of $EFGH$ except for CD is smaller than $-\frac{1}{\varepsilon L}$ up to a constant factor.
First of all, by Lemma 7.1.3, $\mathrm{Im} f_L(z) \lesssim -\frac{1}{\varepsilon L}$ on FG. Next, we consider segments ED and CH. By symmetry, it suffices to study the image of f_L on ED.
Let $z \in ED$. Then, $z = x - i\Delta_n$ with $x \in [\tilde{\lambda}_n, \tilde{\lambda}_n + \Delta_n]$,

$$-\mathrm{Im} f_L(z) = \frac{\tilde{a}_n\Delta_n}{(\tilde{\lambda}_n - x)^2 + \Delta_n^2} + \Delta_n \sum_{k\neq n} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2 + \Delta_n^2}. \tag{7.2.12}$$

According to Lemma 7.1.2, the second term of RHS of (7.2.12) is bounded by $\frac{\Delta_n}{n^2}$. On the other hand, the first term is bigger than $\frac{\tilde{a}_n}{2\Delta_n} \gg \frac{\Delta_n}{n^2}$ since $n > \frac{\eta L}{\ln L}$. Hence,

$$-\mathrm{Im} f_L(z) = \frac{\tilde{a}_n\Delta_n}{(\tilde{\lambda}_n - x)^2 + \Delta_n^2}\left(1 + O\left(\frac{1}{\ln^2 L}\right)\right) \tag{7.2.13}$$

uniformly in $x \in [\tilde{\lambda}_n, \tilde{\lambda}_n + \Delta_n]$. Since the function $\frac{\tilde{a}_n\Delta_n}{(\tilde{\lambda}_n - x)^2 + \Delta_n^2}$ is decreasing in $x \in [\tilde{\lambda}_n, \tilde{\lambda}_n + \Delta_n]$, we infer that $\mathrm{Im} f_L(z)$ is strictly increasing, hence, $f_L(z)$ is injective on ED. Moreover,

$$\mathrm{Im} f_L(z) \asymp -\frac{1}{\Delta_n} \ll -\frac{1}{\varepsilon L}.$$

We consider now $f_L(z)$ on the vertical segment EF where $z = x + iy$ with $x \equiv \tilde{\lambda}_n$ and $-\frac{x_0^2}{\varepsilon L} \le y \le -\Delta_n$. Then,

$$f_L'(z) = -\frac{\tilde{a}_n}{y^2} + \sum_{k\neq n} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - \tilde{\lambda}_n)^2} \cdot \frac{\left(1 + \frac{iy}{\tilde{\lambda}_k - \tilde{\lambda}_n}\right)^2}{\left[1 + \left(\frac{y}{\tilde{\lambda}_k - \tilde{\lambda}_n}\right)^2\right]^2}. \tag{7.2.14}$$

Hence,

$$-\frac{\partial}{\partial y}\mathrm{Im} f_L(z) = -\mathrm{Re}[f_L'(z)] = \frac{\tilde{a}_n}{y^2} - \underbrace{\sum_{k\neq n} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - \tilde{\lambda}_n)^2} \cdot \frac{1 - \left(\frac{y}{\tilde{\lambda}_k - \tilde{\lambda}_n}\right)^2}{\left[1 + \left(\frac{y}{\tilde{\lambda}_k - \tilde{\lambda}_n}\right)^2\right]^2}}_{=:s(y)} \tag{7.2.15}$$

For $k \neq n$, let $u_k := \frac{y^2}{(\tilde{\lambda}_k - \tilde{\lambda}_n)^2}$ and $\psi(u_k) := \frac{1-u_k}{(1+u_k)^2}$. Then, $\psi'(u_k) = \frac{u_k - 3}{(u_k+1)^3}$.
Note that, in the present case,

$$0 < u_k \lesssim \frac{\left(\frac{n}{\varepsilon L}\right)^2 \cdot n^2}{n^2} \lesssim \left(\frac{n}{\varepsilon L}\right)^2.$$

Hence, for any $n \le \frac{\varepsilon L}{C}$ with C large and $k \neq n$, $u_k \in (0, 1/2]$.
Hence, $\psi(u_k)$ is decreasing and $\frac{2}{9} = \psi\left(\frac{1}{2}\right) \le \psi(u_k) \le \psi(0) = 1$.
Therefore, there exists a numeric constant μ s.t.

$$\frac{1}{\mu n^2} \le s(y) \le \frac{\mu}{n^2}. \tag{7.2.16}$$

(7.2.15) and (7.2.16) yield that

$$-\frac{\partial}{\partial y}\mathrm{Im} f_L(z) \ge \frac{c_0(\varepsilon L)^2}{n^4} - \frac{\mu}{n^2} \ge \frac{\mu}{n^2} \tag{7.2.17}$$

for all $n \le \frac{\varepsilon L}{C_1}$ with $C_1 = C_1(\alpha, c_0, \mu)$ large enough.
Hence, in the present case, $\mathrm{Im} f_L(z)$ is decreasing in y. As a result, the function $f_L(z)$ is injective on EF and $|f_L'(z)| \ge |\mathrm{Re}[f'(z)]| \gtrsim \frac{1}{n^2}$.
Besides, on EF,

$$\begin{aligned} \mathrm{Im} f_L(z) &\ge \mathrm{Im} f_L(\tilde{\lambda}_n - i\Delta_n) = \mathrm{Im} f_L(E) \\ &= -\frac{\tilde{a}_n}{\Delta_n} - \Delta_n \sum_{k\neq n} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - \tilde{\lambda}_n)^2 + \Delta_n^2} \asymp -\frac{1}{\Delta_n}; \end{aligned} \tag{7.2.18}$$

and

$$\mathrm{Im} f_L(z) \leq \mathrm{Im} f_L\left(\tilde{\lambda}_n - i\frac{x_0^2}{\varepsilon L}\right) = \mathrm{Im} f_L(F) \tag{7.2.19}$$
$$\asymp -\frac{\tilde{a}_n \varepsilon L}{n^2} - \frac{n^2}{\varepsilon L}\sum_{k\neq n}\frac{\tilde{a}_k}{(\tilde{\lambda}_k - \tilde{\lambda}_n)^2 + \frac{n^4}{(\varepsilon L)^2}} \asymp -\frac{\varepsilon L}{n^2} \lesssim -\frac{1}{\varepsilon L}.$$

By symmetry, we have the same conclusion for the image of GH under f_L.
To sum up, the images of ED, CH, EF, FG, GH under f_L stay below the horizontal $y = -\frac{\tilde{C}}{\varepsilon L}$ in the complex plane with some positive constant $\tilde{C}$. Hence, the claim follows. □

7.2.2 Resonances in $\tilde{\Omega}_n^i$ and Ω^i

First of all, all sides of the rectangle $\tilde{\Omega}_n^i$ are included in horizontal and vertical segments of Ω_n^i. Hence, Lemma 7.2.1 still hold for $\tilde{\Omega}_n^i$. We will prove the existence and uniqueness of rescaled resonances in $\tilde{\Omega}_n^i$.

Theorem 7.2.6. *Pick $n < \frac{\eta L}{\ln L}$ with $\eta > 0$ small. Let $x_0 = \tilde{\lambda}_{n+1} - \tilde{\lambda}_n$ and $\tilde{\Omega}_n^i$ be the rectangle $[\tilde{\lambda}_n + \Delta_n, \tilde{\lambda}_{n+1} - \Delta_n] + i\left[-\frac{x_0^2}{\varepsilon L}, 0\right]$ in Figure 7.2.*
Then, f_L is bijective from $\tilde{\Omega}_n^i$ on $f_L(\tilde{\Omega}_n^i)$ and $|f_L'(z)| \gtrsim \frac{1}{n^2}$. Moreover, there exists a unique rescaled resonance $\tilde{z}_n$ in $\tilde{\Omega}_n^i$ which satisfies

$$|Im \tilde{z}_n| \lesssim \frac{n^2}{\varepsilon L}.$$

Proof of Theorem 7.2.6. Let $\tilde{\gamma}_n$ be the boundary of $\tilde{\Omega}_n^i$.
It's easy to check that, the monotonicity and the estimates we made for the real and imaginary part of $f_L(z)$ and $|f_L'(z)|$ on AB, BC, AD in Lemma 7.2.2 still hold for A_1B_1, A_1D_1, B_1C_1 of the contour $\tilde{\gamma}_n$. Now, we study the image of C_1D_1 under f_L. Let $z = x + iy \in C_1D_1$ with $x \in [\tilde{\lambda}_n + \Delta_n, \tilde{\lambda}_{n+1} - \Delta_n]$ and $y \equiv -\frac{x_0^2}{\varepsilon L}$.
Note that, in the present case, $\Delta_n \geq \frac{x_0^2}{\varepsilon L}$. Hence,$|\tilde{\lambda}_k - x| \geq |y|$ for all k. Moreover, for $k \neq n, n+1$, $|\tilde{\lambda}_k - x| \gtrsim n \gg |y|$. Then, since (7.2.8), we have

$$\mathrm{Re}[f_L'(z)] \gtrsim \sum_{k\neq n,n+1}\frac{1}{(\tilde{\lambda}_k - x)^2} \gtrsim \frac{1}{n^2}$$

for all $z \in C_1D_1$.
Hence, $\mathrm{Re} f_L(z)$ is still strictly increasing in x on C_1D_1. Finally, we compute the magnitude

of $\text{Im} f_L(z)$ on C_1D_1.

$$
\begin{aligned}
-\text{Im} f_L(z) \geq -\text{Im} f_{n,L}(z) &\asymp |y| \left(\frac{1}{(\tilde{\lambda}_n - x)^2 + y^2} + \frac{1}{(\tilde{\lambda}_{n+1} - x)^2 + y^2} \right) \\
&\gtrsim \frac{|y|}{n^2 + y^2} \gtrsim \frac{1}{\varepsilon L}.
\end{aligned} \tag{7.2.20}
$$

Hence, $\text{Im} f_L(z) \lesssim -\frac{1}{\varepsilon L}$. Then, using the same argument as in Lemmata 7.2.2 and 7.2.3, we infer that f_L is bijective from $\tilde{\Omega}^i_n$ on $f_L(\tilde{\Omega}^i_n)$. Moreover, thanks to (7.2.20), we deduce that the point $-\frac{e^{-i\theta(E_0)}}{L}$ belongs to $f_L(\tilde{\Omega}^i_n)$. Hence, there exists a unique rescaled resonance in $\tilde{\Omega}^i_n$. □

Finally, we show that, there are no rescaled resonances in $\mathcal{R}^i$.

Theorem 7.2.7. *Pick $0 < \delta_1 < \tilde{\lambda}_0$ and ε small, fixed numbers.*
Let $E_0 \in (-2, 2)$ be the left endpoint of the ith band B_i of $\Sigma_{\mathbb{Z}}$. Let $(\lambda^i_\ell)^{n_i}_{\ell=0}$ be (distinct) eigenvalues of H_L in B_i.
Let Ω^i be the rectangle $[0, \tilde{\lambda}^i_0 - \delta_1] + i \left[-\frac{1}{\varepsilon L}, 0\right]$ in Figure 7.3.
Then, f_L is bijective from Ω^i on $f_L(\Omega^i)$ and $|f'_L(z)| \geq c > 0$. Moreover, $f_L(\Omega^i)$ does not contain the point $-\frac{e^{-i\theta(E_0)}}{L}$, hence, there are no resonances in Ω^i.

Proof of Theorem 7.2.7. Note that if E_0 is an eigenvalue of H_L for L large i.e., $E_0 = \lambda^i_0$, we have $\mathcal{R}^i = \emptyset$. Let's assume now that E_0 is not an eigenvalue of H_L for L large.
First of all, we will check that the rescaled resonance equation (7.0.1) in Ω^i can be replaced by $f_L(z) = -\frac{e^{-i\theta(E_0)}}{L}$.
Indeed, along the segment A_3B_3, $f_L(z)$ is real. Along B_3C_3, $|f_L(z)|$ is big. Along C_3D_3, $|\text{Im} f_L(z)|$ is big. Hence, to prove Lemma 7.2.1 for Ω^i, it suffices to check that

$$|f_L(z)| \gtrsim \frac{1}{\varepsilon L} \text{ on } A_3D_3.$$

Put $z = iy \in A_3D_3$ with $0 \geq y \geq -\frac{1}{\varepsilon L}$. Assume that $\lambda^i_k \geq E_0 + 2\varepsilon_1$ with $\varepsilon_1 \asymp \varepsilon^2$ for all $k > \varepsilon L$ and $\lambda^i_k \in B_i$. Then, by Lemma 7.1.1, we have

$$\text{Re} f_L(z) = \sum_{k=0}^{\varepsilon L} \frac{\tilde{a}^i_k \tilde{\lambda}^i_k}{(\tilde{\lambda}^i_k)^2 + y^2} + O\left(\frac{1}{\varepsilon_1 L}\right). \tag{7.2.21}$$

For any $\lambda_k \notin B_i$, $|\tilde{\lambda}_k| = L^2|\lambda_k - E_0| \gtrsim L^2$. On the other hand, if $\lambda_k \in B_i$, we have $\lambda_k \neq E_0$ and $|\tilde{\lambda}_k| = L^2|\lambda_k - E_0| \geq L^2|\lambda^i_0 - E_0| \gtrsim 1$. Hence, $|\tilde{\lambda}_k| \geq \frac{1}{C} \gg |y|$ for all λ_k. On the other hand, $\tilde\lambda_k^i > 0$ for all $\lambda_k^i \in B_i$. Consequently,

$$\sum_{k\le\varepsilon L} \frac{\tilde a_k^i \tilde\lambda_k^i}{(\tilde\lambda_k^i)^2 + y^2} \asymp \sum_{k\le\varepsilon L} \frac{\tilde a_k^i}{\tilde\lambda_k^i} \asymp \sum_{k=1}^{\varepsilon L} \frac{1}{k^2} \asymp 1. \tag{7.2.22}$$

The estimates (7.2.21) and (7.2.22) yield $\mathrm{Re} f_L(z) \asymp 1$ on A_3D_3. Hence, Lemma 7.2.1 holds true for Ω^i.

Next, we will study the image of the contour $A_3B_3C_3D_3$ under f_L.

On A_3B_3, $f_L(z)$ is real and strictly increasing. Hence

$$\sum_{k=0}^{L} \frac{\tilde a_k}{\tilde\lambda_k} \le f_L(z) \le \sum_{k=0}^{L} \frac{\tilde a_k}{\tilde\lambda_k - \tilde\lambda_0^i + \delta_1} \tag{7.2.23}$$

where C is a positive constant.

Thanks to Lemma 7.1.1 and (7.2.22), it is easy to see that $\sum_{k=0}^{L} \frac{\tilde a_k}{\tilde\lambda_k} \asymp 1$. Similarly, we have

$$\sum_{k=0}^{L} \frac{\tilde a_k}{\tilde\lambda_k - \tilde\lambda_0^i + \delta_1} = \frac{\tilde a_0^i}{\delta_1} + \sum_{k=1}^{\varepsilon L} \frac{\tilde a_k^i}{\tilde\lambda_k^i - \tilde\lambda_0^i + \delta_1} + O\left(\frac{1}{\varepsilon_1 L}\right) \asymp \frac{1}{\delta_1}. \tag{7.2.24}$$

Hence, $f_L(z) \asymp 1$ on the interval A_3B_3.

Next, we consider the segment A_3D_3. Since $|\tilde\lambda_k| \gtrsim 1 \gg |y|$ for all $\lambda_k \in \Sigma_{\mathbb{Z}}$, we have

$$\frac{\partial}{\partial y}\mathrm{Im} f_L(z) = \mathrm{Re}[f_L'(z)] = \sum_{k=0}^{L} \frac{\tilde a_k}{\tilde\lambda_k^2} \cdot \frac{1 - \frac{y^2}{\tilde\lambda_k^2}}{\left[1 + \frac{y^2}{\tilde\lambda_k^2}\right]^2} \gtrsim \sum_{k=0}^{L} \frac{\tilde a_k}{\tilde\lambda_k^2} \tag{7.2.25}$$

where $z = iy$ with $0 \ge y \ge -\frac{1}{\varepsilon L}$.

We will show that, for all $z = x + iy \in \Omega^i$,

$$\sum_{k=0}^{L} \frac{\tilde a_k}{(\tilde\lambda_k - x)^2 + y^2} \asymp 1. \tag{7.2.26}$$

Indeed, since $|\tilde\lambda_k - x| \gg |y|$ for all $z = x+iy \in \Omega^i$ and all $\lambda_k \in \Sigma_{\mathbb{Z}}$, we have $(\tilde\lambda_k - x)^2 + y^2 \asymp (\tilde\lambda_k - x)^2$. Then, argument as in (7.2.22), (7.2.24), we have (7.2.26) follow.

Consequently, $\mathrm{Im} f_L(z)$ is strictly increasing on A_3D_3 and $|f_L'(z)| \gtrsim 1$ on A_3D_3.

Now, we give estimates on the real and imaginary parts of $f_L(z)$ on A_3D_3.

$$0 \ge \mathrm{Im} f_L(z) \ge \mathrm{Im} f_L(D_3) = \mathrm{Im} f_L\left(-\frac{i}{\varepsilon L}\right) \asymp -\frac{1}{\varepsilon L}\left(\sum_{k=0}^{L} \frac{\tilde a_k}{\tilde\lambda_k^2}\right) \asymp -\frac{1}{\varepsilon L}. \tag{7.2.27}$$

Besides, as we proved before, $\mathrm{Re} f_L(z) \asymp 1$ on A_3D_3.
Similarly, we have the same conclusion for $f_L(z)$ on B_3C_3.
Finally, we study f_L on C_3D_3. Let $z \in C_3D_3$, $z = x + iy$ where $x \in [0, \tilde{\lambda}_0^i - \delta_1]$ and $y \equiv -\frac{1}{\varepsilon L}$.
Using the equation (7.2.8), we can check easily that $\mathrm{Re}[f'_L(z)]$ is bigger than a positive constant on C_3D_3. Hence, $\mathrm{Re} f_L(z)$ is strictly increasing in x. Consequently, on C_3D_3,

$$1 \asymp \mathrm{Re} f_L(D_3) \leq \mathrm{Re} f_L(z) \leq \mathrm{Re} f_L(C_3) \asymp 1. \tag{7.2.28}$$

Finally, we compute the magnitude of $\mathrm{Im} f_L(z)$. For $z = x + iy \in C_3D_3$, (7.2.26) yield

$$\mathrm{Im} f_L(z) = y \left(\sum_{k=0}^{L} \frac{\tilde{a}_k}{(\tilde{\lambda}_k - x)^2 + y^2} \right) \asymp -\frac{1}{\varepsilon L}. \tag{7.2.29}$$

To sum up, f_L is bijective from Ω^i to $f_L(\Omega^i)$ and $|f'_L(z)| \gtrsim 1$ for all $z \in \Omega^i$. Moreover, there exists a positive constant c such that $\mathrm{dist}(0, f_L\left(\Omega^i\right)) \geq c$. Hence, $-\frac{e^{-i\theta(E_0)}}{L} \notin f_L(\Omega^i)$ which implies that there are no resonances in Ω^i. $\square$

APPENDIX A

Determinant of matrices A_0, A_1 in Chapter 4

Compute the determinant of matrix A_0 *in (4.2.31).* Put $A_0 = (a_{ij})$, we'll give here some details of computing the determinant of A_0 by hand (A mathematical software like Maple or Mathematica might be useful for checking the final result of this computation).

First, expand this determinant by its sixth and last column and then by its first column to get

$$|\det A_0| = \omega_{n-2} |\det B_0|$$

where B_0 is the 7×7 matrix defined by

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ -\omega_{n-1} & \omega_{n-1} + \omega_n - E & -\omega_n & 0 & 0 & 0 & 0 \\ 0 & \dfrac{E}{E'} & 0 & 0 & 0 & -1 & 0 \\ 0 & -\omega_n & \omega_n + \omega_{n+1} - E & -\omega_{n+1} & 0 & 0 & 0 \\ 0 & \omega_n & \omega_{n+1} - \omega_n & -\omega_{n+1} & 0 & 0 & E' \\ \dfrac{E}{E'} & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Now, we compute the determinant of B_0.

Take the sixth row minus the fifth row and take the first row minus the last row. Next, multiply the second row by E' and take it minus the sixth row. Finally, expand the determinant of B_0 by the forth, the fifth and the last column to get

$$|\det A_0| = \omega_{n-2}\omega_{n+1}|\det C_0|$$

where C_0 is the 4×4 matrix defined by

$$\begin{pmatrix} 1-\dfrac{E}{E'} & -1 & 0 & 1 \\ 0 & E'-2\omega_n & -E-E'+2\omega_n & -E' \\ -\omega_{n-1} & \omega_{n-1}+\omega_n-E & -\omega_n & 0 \\ 0 & \dfrac{E}{E'} & 0 & -1 \end{pmatrix}.$$

Finally, by an explicit computation for the determinant of C_0, we obtain that

$$|\det A_0| = \frac{4E}{E'}|E-E'|\omega_{n-2}\omega_{n+1}\left|\omega_n - \frac{E'+E}{4}\right|.$$

□

Compute the determinant of matrix A_1 in (4.2.32). The determinant of A_1 can be computed as follows: First, expand this determinant by its sixth and last column and then by its fifth and first column to get

$$|\det A_1| = \omega_{n-2}\omega_{n+1}|\det B_1|$$

where B_1 is the 6×6 matrix defined by

$$\begin{pmatrix} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ -\omega_{n-1} & \omega_{n-1}+\omega_n-E & -\omega_n & 0 & 0 & 0 \\ \omega_{n-1} & \omega_n-\omega_{n-1} & -\omega_n & 0 & -E' & 0 \\ 0 & 0 & \dfrac{E}{E'} & 0 & 0 & -1 \\ -\dfrac{E}{E'} & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Second, take the first row of matrix B plus its last row and take the second row plus the fifth row, then expand the determinant of B_1 by its forth and sixth column to obtain

$$|\det A_1| = \omega_{n-2}\omega_{n+1}|\det C_1|$$

where C_1 is the following 4×4 matrix

$$\begin{pmatrix} 1 - \dfrac{E}{E'} & -1 & 0 & -1 \\ 0 & 1 & -1 + \dfrac{E}{E'} & -1 \\ -\omega_{n-1} & \omega_{n-1} + \omega_n - E & -\omega_n & 0 \\ \omega_{n-1} & \omega_n - \omega_{n-1} & -\omega_n & -E' \end{pmatrix}.$$

Finally, by an explicit computation for the determinant of the matrix C_1, we infer that

$$|\det A_1| = \frac{4E}{E'} \omega_{n-2}\omega_{n+1} \left| \omega_{n-1}\omega_n - \frac{(E - E')^2}{4} \right|.$$

□

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